University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Topology from Physics Winter 2021
Assignment 4 – Solutions

Due 12:30pm Wednesday February 3, 2021

Thanks in advance for following the guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Subdivision invariance.** Subdivide a complex C made of a single 3-simplex (and its sub-simplices) into a complex $\hat{C}$ with four 3-simplices. Show that the complex $\hat{C}/C$ has no homology.

   This subdivision adds a single point p in the center. It adds four new edges running from p to one of the original corners p_i. There are six new triangular faces, which we can label by their three vertices; each of them contains two original points p_i, p_j and p, so we could call them w_{pij}. There are four new volumes v_i. I label them by which of the original corners p_i they *do not* touch.

   The generators of the quotient complex are then:

$$C'_0 = \text{span}\{p\}, C'_1 = \text{span}\{y_{pi}, i = 1..4\}, \tag{1}$$
$$C'_2 = \text{span}\{w_{pij}, i \neq j = 1..4\}, C'_3 = \text{span}\{v_i, i = 1..4, \sum_i v_i = v = 0 \text{ mod } C\}. \tag{2}$$

   The boundary maps are then

$$\partial y_{pi} = p - p_i = p \text{ mod } C, \partial w_{pij} = y_{pi} - y_{pj} \pm y_{p_i p_j} = y_{pi} - y_{pj} \text{ mod } C, \tag{3}$$
$$\partial v_i = \epsilon_{ijkl} \left(w_{pjk} + w_{pkl} + w_{pli} + w_{p_j p_k p_l} \right) = \epsilon_{ijkl} \left(w_{pjk} + w_{pkl} + w_{pli} \right) \text{ mod } C.$$

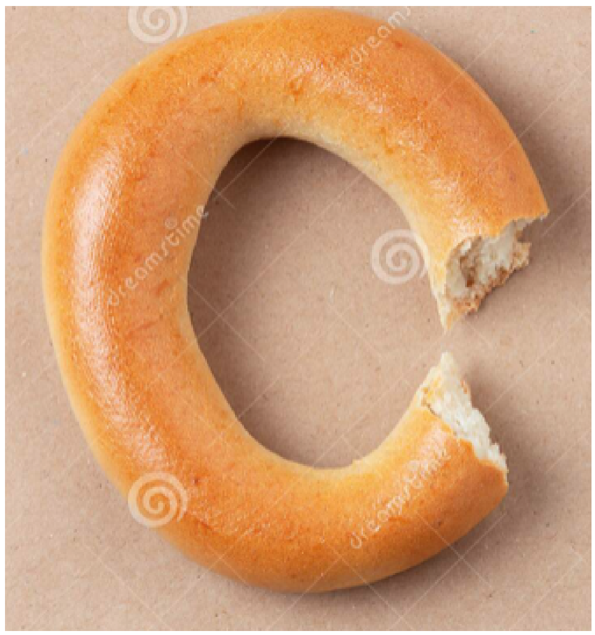
   So the complex is
$$0 \to \mathbb{Z}^4 \xrightarrow{L} \mathbb{Z}^6 \xrightarrow{M} \mathbb{Z}^4 \xrightarrow{N} \mathbb{Z} \to 0.$$

   Ordering the faces as (12,13,14,23,24,34),

$$L = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}, \quad N = (1,1,1,1).$$

You can check that $ML = 0$ and $NM = 0$. The 4×6 matrix L has rank 3 (its singular values are $(2, 2, 2, 0)$) and its 1d kernel is spanned by $\sum_i v_i = 0$, so there is no homology at degree 3. The 6×4 matrix M has a 3d kernel spanning the complement of the 3d image of L (its singular values are also $(2, 2, 2, 0)$), so there is no homology at degree 3. The 4×1 matrix N has rank 1 killing all the rest of the homology.

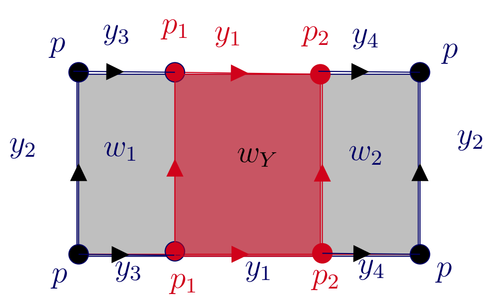
2. **Relative homology.** Take a torus X (like the surface of a bagel) and take a bite Y out of it. Choose the bite so that both Y and $X \setminus Y$ are annuli.



Choose a cell decomposition of X so that Y is closed (meaning that the boundaries of all cells in Y are also in Y). (This means that $X \setminus Y$ has rough boundary conditions and Y has smooth boundary conditions.) Compute $H_\bullet(X, \mathbb{Z})$, $H_\bullet(Y, \mathbb{Z})$, $H_\bullet(X/Y, \mathbb{Z})$. Show that your answers are consistent with the long exact sequence.

[Note that a more common situation is where one uses the long exact sequence to learn $H_\bullet(X, \mathbb{Z})$ from $H_\bullet(Y, \mathbb{Z})$ and $H_\bullet(X/Y, \mathbb{Z})$.]

Here is a cell decomposition which does the job:



The red bits are Y. The complex for Y is

$$0 \to \mathbb{Z} \xrightarrow{(0,1,-1)} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}} \mathbb{Z}^2 \to 0$$

which has homology $0, \mathbb{Z}, \mathbb{Z}$ in degree $0, 1, 2$. The complex for X/Y is

$$0 \to \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} -1 & 1 & 0 \end{pmatrix}} \mathbb{Z} \to 0.$$

which has homology $\mathbb{Z}, \mathbb{Z}, 0$ in degree $0, 1, 2$. This is indeed consistent with the long exact sequence:

$$\begin{tikzcd}
 & H_\bullet(Y) \arrow[r] & H_\bullet(X) \arrow[r] & H_\bullet(X/Y) \\
0: & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[dll, "\partial_\star"] \\
1: & \mathbb{Z} \arrow[r] & \mathbb{Z}^2 \arrow[r] & \mathbb{Z} \arrow[dll, "\partial_\star"] \\
2: & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

3. **Subdivision invariance and entanglement renormalization.**

   (a) Verify that conjugation by the control-X gate

$$\mathsf{CX} \equiv P_C(0) \otimes \mathbb{1}_T + P_C(1) \otimes X_T$$

(with $P_C(0) = |0\rangle\langle 0|_C = \frac{1}{2}(1 + Z_C)$, $P_C(1) = |1\rangle\langle 1|_C = \frac{1}{2}(1 - Z_C)$), accomplishes the operations $(\mathcal{O} \leftrightarrow \mathsf{CX}\mathcal{O}\mathsf{CX})$

$$\begin{aligned}
1_C Z_T &\leftrightarrow Z_C Z_T \\
1_C X_T &\leftrightarrow 1_C X_T \\
Z_C 1_T &\leftrightarrow Z_C 1_T \\
X_C 1_T &\leftrightarrow X_C X_T
\end{aligned}$$

One way to do it is just to use $XZ = -ZX$ and expand out the definitions:

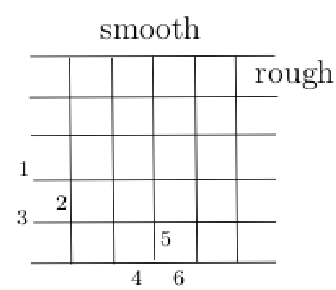
$$\begin{aligned}
\mathsf{CX}\,(1_C \otimes Z_T)\,\mathsf{CX} &= (P_C(0) \otimes 1 + P_C(1) \otimes X)\,(1_C \otimes Z_T)\,\mathsf{CX} && (4) \\
&= (1_C \otimes Z_T)\,(P_C(0) \otimes 1 - P_C(1) \otimes X)\,\mathsf{CX} && (5) \\
&= (1_C \otimes Z_T)\,(P_C(0) \otimes 1 - P_C(1) \otimes 1) && (6) \\
&= (1_C \otimes Z_T)\,(P_C(0) - P_C(1)) \otimes 1 = Z_C \otimes Z_T. && (7)
\end{aligned}$$

In the penultimate line we used $P_C(i)P_C(j) = \delta_{ij}P_C(i)$ are orthogonal projectors.

(b) Find the Hamiltonians resulting from the operations described in the lecture notes which add a new plaquette or add a new vertex to the cell complex. (Note that some arrows were reversed in the vertex-addition-circuit in a previous version of the lecture notes.) Show that each one has the same topological groundstate degeneracy as the toric code on the new cell complex.

The tricky bit is that the X term for the new link becomes a plaquette term for the lower right triangle, but there is no plaquette term for the upper left triangle. Instead, the plaquette operator for the original plaquette is invariant under conjugation by the control-Xs. The product of this with the lower right triangle is the upper left triangle, so the *stabilizer algebra* (the algebra generated by the (commuting) terms in the hamiltonian) is the same as the original toric code. A similar thing happens with the star terms.

4. **A topological qubit.** Consider the toric code on this cell complex:



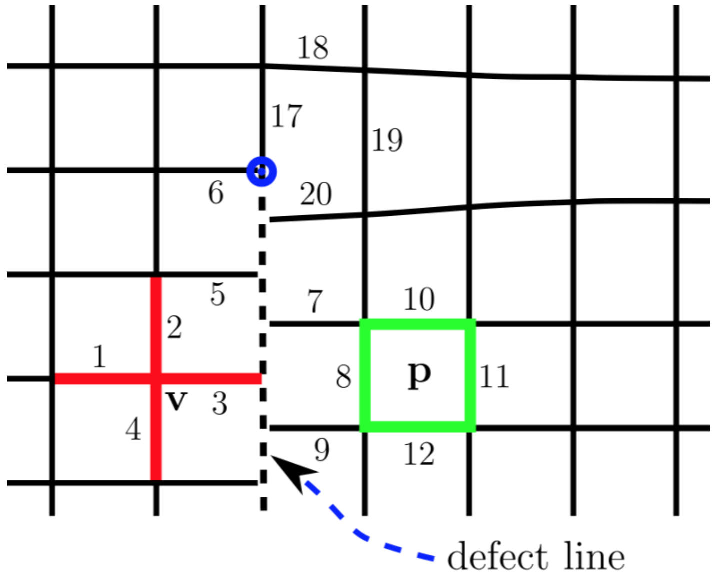
Recall that rough boundary conditions means that plaquette terms get truncated, such as the term $-X_1X_2X_3$, while smooth boundary conditions mean that star terms get truncated, such as the term $-Z_4Z_5Z_6$.

Show that there is a two-dimensional space of groundstates. A good way to do this is using the algebra of string operators which terminate at the various components of the boundary without creating excitations.

An electric string $W_C = \prod_{\ell \in C} X_\ell$ can end without creating excitations on the rough boundaries. A magnetic string $V_{\hat{C}} = \prod_{\ell \perp \hat{C}} Z_\ell$ can end without creating excitations on the smooth boundaries. $VW = -WV$, and they commute with the toric code Hamiltonian, so there is a pair of degenerate groundstates.

5. **Duality wall.** [Bonus problem] Show that the following hamiltonian realizes a

duality wall in the toric code.



What this means is that when crossing the wall, an e particle turns into an m particle and vice versa. (More precisely, there is a string operator which transports an e particle to the wall, and can be completed by a string operator transporting an m particle away from the wall, without creating any excitations.)

To be more precise about the figure: the dotted line carries no degrees of freedom. The terms in the hamiltonian along the wall are of the form

$$H = \ldots - X_2 X_5 X_3 Z_7 - Z_3 X_7 X_8 X_9$$

and there is a term at the end of the wall (the little blue circle) of the form $-Z_6 Y_{17} X_{18} X_{19} X_{20}$. Show that these terms commute with each other and all the usual star and plaquette terms, such as $A = Z_1 Z_2 Z_3 Z_4$ and $B = X_8 X_{10} X_{11} X_{12}$.

What can you say about the end of the duality wall (the little blue circle)?

The figure (and the result) comes from this paper by Kitaev and Kong.

A string operator which is the usual W operator made of Xs along the curve on the left (and hence transports an e particle) can be connected to a V operator made of a string of Zs crossing the curve on the right (and hence transports an m particle) without violating any of the terms in the hamiltonian. And vice-versa.

The endpoint of the duality wall can absorb a fermion (the boundstate of e and m particles).