University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Topology from Physics Winter 2021 Assignment 5

Due 12:30pm Wednesday February 10, 2021

Thanks in advance for following the guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Hodge star and adjoint of d.** Consider the inner product on (real) p-forms on a manifold without boundary $\mathcal{M}$

$$\langle B, A\rangle \equiv \int_{\mathcal{M}} (\star B) \wedge A.$$

Show (using integration by parts and $\star^2 = (-1)^k$) that the adjoint of the exterior derivative can be written as

$$d^\dagger = s \star d \star$$

where s is a sign depending on p and $\dim \mathcal{M}$.

Bonus problem: Find s.

2. **Supersymmetric harmonic oscillator.** Consider the quantum mechanical system with Hamiltonian

$$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 x^2 + \frac{1}{2}\omega[\bar{\psi}, \psi].$$

(a) Using your knowledge of the ordinary harmonic oscillator, construct the spectrum. Consider both signs of ω.

(b) Compute the thermal partition function at temperature β, $Z(\beta) = \mathrm{tr} e^{-\beta H}$, and the Witten index $\mathrm{tr}(-1)^F = \mathrm{tr}(-1)^F e^{-\beta H}$.

3. **Supersymmetry in $D = 0$ and localization.**

In this problem, we consider the 'action'

$$S[x, \psi, \bar{\psi}] = \frac{1}{2}\left(\partial_x h(x)\right)^2 + a\bar{\psi}\psi\partial^2 h(x)$$

for a field theory in $D = 0$ dimensions and the associated 'partition function'

$$Z[h] \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx d\psi d\bar{\psi} e^{-S[x,\psi,\bar{\psi}]}.$$

Here $\psi, \bar{\psi}$ are independent grassmann variables, but otherwise this is just an ordinary finite-dimensional integral.

(a) Show that the action S is supersymmetric (for some choice of constant a), in the sense that it is invariant under the transformation

$$\delta_\epsilon x = \epsilon\psi - \bar{\epsilon}\bar{\psi}, \quad \delta_\epsilon \psi = \bar{\epsilon}\partial h, \quad \delta_\epsilon \bar{\psi} = \epsilon \partial h.$$

$\epsilon, \bar{\epsilon}$ are independent grassmann variables.

(b) Prove the supersymmetry Ward identity

$$0 = \left\langle \delta_\epsilon g(x, \psi, \bar{\psi}) \right\rangle \tag{1}$$

where

$$\langle \mathcal{O} \rangle \equiv \frac{1}{\sqrt{2\pi}} \int dx d\psi d\bar{\psi} e^{-S[x,\psi,\bar{\psi}]} \mathcal{O}$$

for any function $\mathcal{O}$ of the dynamical variables, and g is a function with good enough behavior at $|x| \to \infty$.

[Hint: there is no boundary of the integration region.]

[Second hint: Very generally, if the action and integration measure are invariant under some symmetry then

$$\langle \delta g \rangle \equiv \int \delta g e^{-S} = 0$$

where δg is the variation of g under the symmetry (as long as g behaves well at the boundaries of the integration region). This follows from changing variables in the integral. Consider, for example, the case $\int_{\mathbb{R}^2} dx dy e^{-S[x,y]}$, with $S[x, y] = x^2 + y^2$ and $\delta x = y, \delta y = -x$ (rotations).]

(c) By choosing $g(x, \psi, \bar{\psi}) = \partial\rho(x)\psi$ (for some function $\rho(x)$) in the Ward identity (1), show that changing $h(x) \to h(x) + \rho(x)$ in the action does not change the partition function Z, *i.e.* $Z[h + \rho] = Z[h]$, for infinitesimal ρ.

Now the hard part. In the next few parts, we wish to show that the integral $Z[h]$ is localized to loci where the supersymmetry variation of the fermions is zero. This is called the *supersymmetric localization principle*. In this case this means $0 = \delta\psi \propto \partial_x h(x) \equiv h'(x)$, critical points of h.

(d) Argue that if $h'(x) = 0$ has no real solutions, then $Z[h] = 0$. Hint: change integration variables to $\tilde{x} = x - \bar{\psi}\psi/h'(x)$.

(e) Hence the integral gets contribution only from the neighborhood of critical points of h. Taylor expand to second order in h near such a critical point and evaluate its contribution.

(f) Add up the results and conclude that the partition function is an integer.

(g) Conclude using the result of part (3c) that $Z[h] = 0$ if h is polynomial of odd degree, while $Z[h] = \pm 1$ if h is a polynomial of even degree.

(h) Argue that the Witten index for the supersymmetric SHO of problem 2 reduces to $Z[h = \omega x^2]$ in the limit $\beta \to 0$.

(i) [More optional] Generalize the results of this problem this problem to n variables $x_i, \psi_i, \bar{\psi}_i$, and a superpotential h which depends on all of them.