University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Topology from Physics Winter 2021
Assignment 6

Due 5pm Friday February 19, 2021

Thanks in advance for following the guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Euler characteristics of spheres.** Consider the N-dimensional unit sphere

$$S^N \equiv \{(x_0, \cdots, x_N) | x_0^2 + x_1^2 + \cdots x_N^2 = 1\}.$$

Let k be the isometry taking $x_N \to -x_N$. Show that the supersymmetric non-linear sigma model on S^N has

$$\mathrm{tr}(-1)^F = 1 + (-1)^N, \qquad \mathrm{tr}(-1)^F K = 1 - (-1)^N$$

where K is the unitary operation implementing k and commuting with supersymmetry. Hint: use the Morse function $h = x_0$.

2. **Cohomology of $\mathbb{CP}^N$.** [Bonus problem] Complex projective space can be defined as

$$\mathbb{CP}^N \equiv \{(z_0, z_1 \cdots z_N)\}/(z \sim \lambda z), \lambda \in \mathbb{C}^\star \equiv \mathbb{C} \setminus 0.$$

Compute the Witten index for the supersymmetric NLSM with this target space.

Construct the groundstates of the supersymmetric non-linear sigma model on this space.

An alternative definition is

$$\mathbb{CP}^N = \{(z_0, z_1 \cdots z_N), \sum_{i=0}^{N} |z_i|^2 = 1\}/(z \sim e^{\mathbf{i}\theta} z)$$

where we can regard the equivalence relation as a gauge redundancy.

3. **Integral formula for the Euler character from the path integral**

In this problem, we use the independence of Witten index $\mathrm{tr}(-1)^F e^{-\beta H}$ from β to derive an expression for the Euler number of a manifold in terms of an integral involving the Riemann curvature tensor over the manifold.

Consider the limit $\beta \to 0$ of the path-integral representation of the Witten index, and argue that the finite-action field configurations contributing to the path-integral localize to constant modes independent of time. This reduces the path-integral to a zero-dimensional QFT, involving an integration over the manifold and some grassmann variables. (This could also be derived using the localization principle and the supersymmetry transformation of the fermionic fields). Show that the fermionic integration brings down Riemann curvature terms from the quartic fermionic term in the action leading to the desired integral over the manifold.

Show in this way that the Euler character vanishes for odd-dimensional manifolds.

Bonus problem: Find an integral formula for the Hirzebruch index. This requires first finding a path integral representation.