University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Topology from Physics Winter 2021 Assignment 8 – Solutions

Due 5pm Friday March 5, 2021

Thanks in advance for following the guidelines on hw01. Please ask me by email if you have any trouble.

1. Cech cohomology brainwarmer. Check that the Cech coboundary operator δ is nilpotent: $\delta^2 = 0$.

$$(\delta^2 \omega)_{\alpha_0 \cdots \alpha_{p+2}} = \sum_i (-1)^i (\delta \omega)_{\alpha_0 \cdots \widehat{\alpha_i} \cdots \alpha_{p+2}} \tag{1}$$

$$= \sum_{j < i} (-1)^{i+j} \omega_{\alpha_0 \cdots \widehat{\alpha_j} \cdots \widehat{\alpha_i} \cdots \alpha_{p+2}} + \sum_{j > i} (-1)^{i+j-1} \omega_{\alpha_0 \cdots \widehat{\alpha_i} \cdots \widehat{\alpha_j} \cdots \alpha_{p+2}}$$
 (2)

$$=0. (3)$$

2. Euler-Poincaré theorem for Cech cohomology.

Let X be a manifold with a finite good cover. Let β_p is the number of non-empty (p+1)-fold intersections $U_{\alpha_0\cdots\alpha_p}$. Show that the Euler character is

$$\chi(X) = \sum_{p} (-1)^p \beta_p.$$

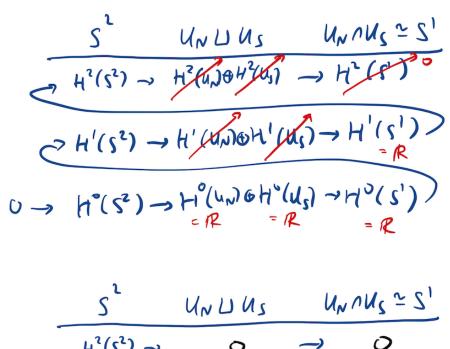
Since the cohomology groups over \mathbb{R} can be computed by a complex with dimensions β_p , this follows by the same proof as the Euler-Poincaré theorem for homology that we proved earlier.

3. Cech cohomology example.

Convince yourself that the computation of the Cech cohomology of the 2-sphere (with arbitrary coefficients) using the good cover described in lecture is the same as the computation of the homology of the tetrahedron cell complex.

4. Homology of spheres.

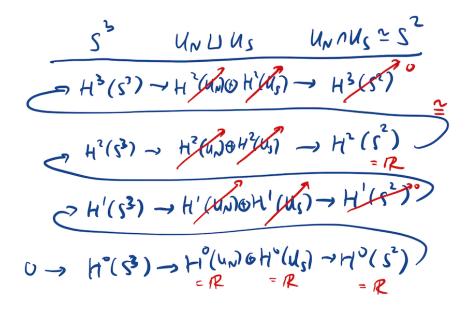
(a) Use the Mayer-Vietoris sequence to compute $H^q(S^n)$ using the open cover $S^n = U_N \cup U_S$, where $U_N(U_S)$ is the complement of the north (south) pole. Start with S^2 and work your way up.



$$\frac{S^{2}}{H^{2}(S^{2})} \rightarrow 0 \rightarrow R$$

$$\Rightarrow H^{2}(S^{2}) \rightarrow R \oplus R \rightarrow R$$

$$0 \rightarrow H^{2}(S^{2}) \rightarrow R \oplus R \rightarrow R$$



(b) Consider the sphere $S^n(r) = \{\sum_{i=0}^n x_i^2 = r^2\} \subset \mathbb{R}^{n+1}$. Show that

$$\omega \equiv \frac{1}{r} \sum_{i=0}^{n} (-1)^{i} x_{i} dx_{0} \wedge \cdots \widehat{dx_{i}} \wedge dx_{n}$$

(the one with the hat is omitted) is not exact by integrating it over S^n . It is a generator of $H^n(S^n)$. [Hint: $dr \wedge \omega = dx_0 \wedge \cdots dx_n$ is the volume form on \mathbb{R}^{n+1} .]

 $S^n(r) = \partial B^{n+1}(r)$ so Stokes says

$$\int_{S^n(r)} \omega = \int_{B^{n+1}(r)} d\omega = \frac{n+1}{r} \int_{B^{n+1}(r)} dx_0 \wedge \cdots dx_n = \partial_r \left(\operatorname{vol}(B^{n+1}(r)) \right) \neq 0.$$

Restricted to $S^n(r)$, however, $d\omega = 0$ since there are no n + 1-forms on S^n . So on $S^n(r)$ ω is closed but not exact (if it were exact on S^n it couldn't integrate to something nonzero).