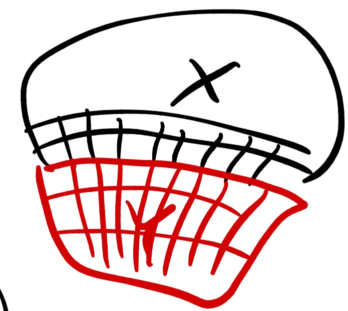
Last time: • Gapped b.c. on Toric code

Rough b.c.s allow strings to end at bdy

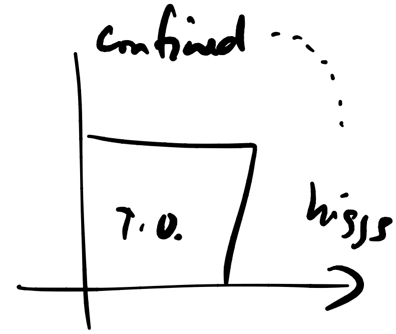
$\longleftrightarrow$ relative homology $H(X/Y, A)$

• idea: e-particles condense in Y

(Y is in Higgs phase)



(also:
Smooth bc: m-particles condense in Y)



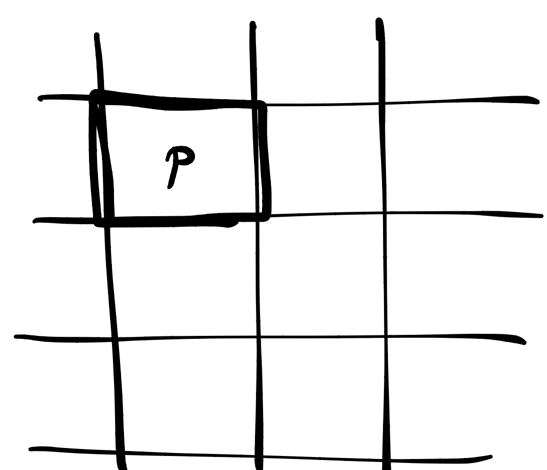
• Possible gapped boundary conditions says a lot about the T.O.

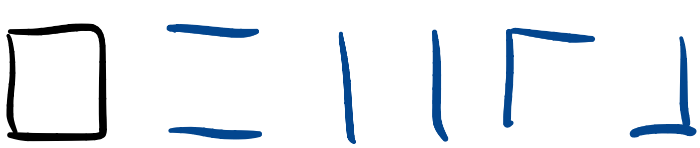
1.7 Duality

Let's write $|gs\rangle$ in the X-basis.

$B_p = 1 \Longrightarrow \prod_\ell x_\ell = 1$

$= \prod_{\ell \in \partial p} X_\ell$

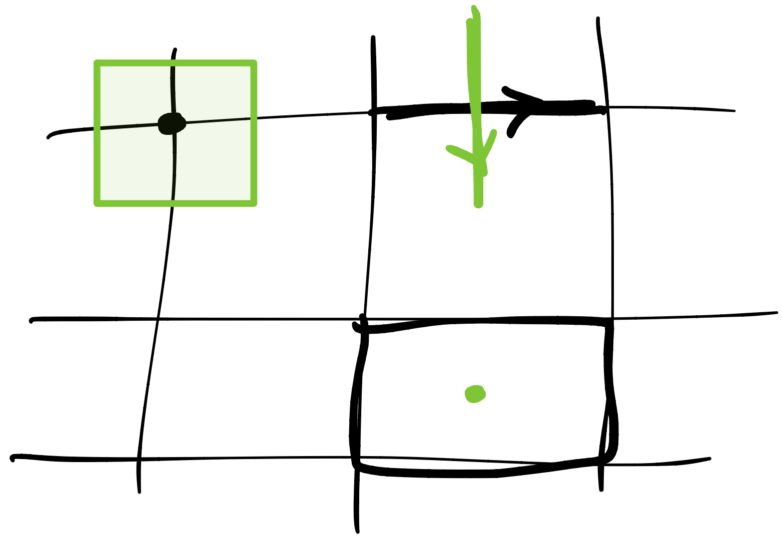


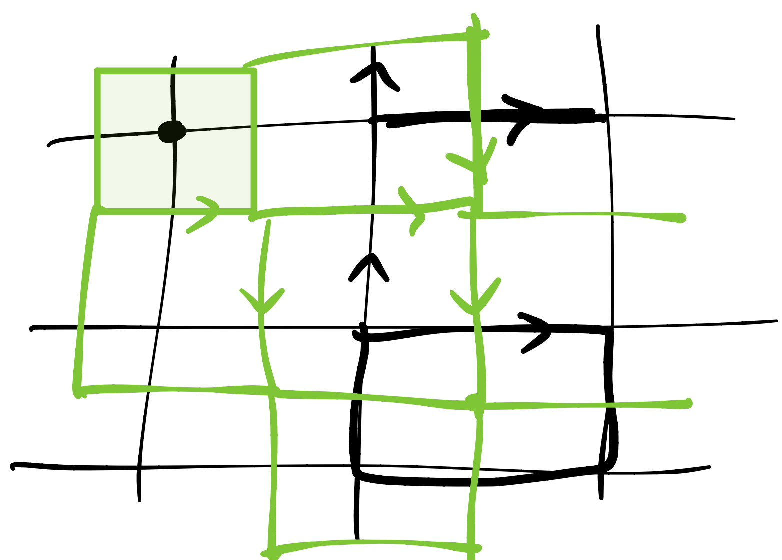


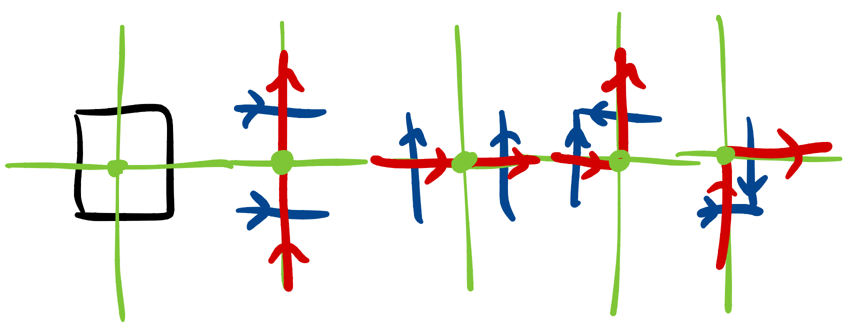
Dual lattice $\Delta^\vee$:
for a d-dim'l lattice.

$$\Delta_p \equiv \Delta^\vee_{d-p} \qquad (\text{\textbackslash vee.})$$

$$\boxed{d=2}$$







$B_p = 1 \iff$ red strings are closed at $p^\vee \in \Delta^\vee_0 = \Delta_2$

A local unitary takes $X \leftrightarrow Z$

$$\begin{cases} H Z H^\dagger = X \\ H X H^\dagger = Z \end{cases}$$

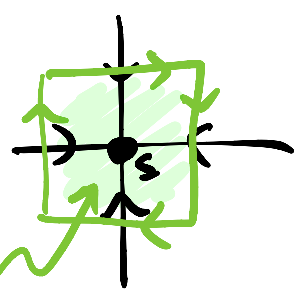
$H^\dagger = H$, $H^\dagger H = 1$, $H^2 = 1$

$$H = |0\rangle\langle +| + |1\rangle\langle -| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Hadamard gate.

$A = \mathbb{Z}_2$:

$$\prod_{\ell \in v(s)} Z_\ell = \prod_{\ell \in \partial^\vee(s^\vee)} Z_\ell .$$



$s^\vee \in \Delta_2^\vee$

A local unitary $\left(\bigotimes_\ell H_\ell\right)$ + relabelling takes TC on Δ to TC on $\Delta^\vee$.

$\mathbb{Z}_N$: $H = \frac{1}{\sqrt{N}}$ (character table of $\mathbb{Z}_N$)

$$\begin{cases} H Z H^\dagger = X \\ H X H^\dagger = Z \end{cases}$$

eg $$H_{(3)} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}$$

$$\chi_a(g_\alpha) \equiv \mathrm{tr}_{R_a} \, \mathcal{U}(g_\alpha) = \mathrm{tr}_{R_a} \, \mathcal{U}(h g_\alpha h^{-1})$$

irrep of G

conjugacy class α

Z basis : Conjugacy classes of A

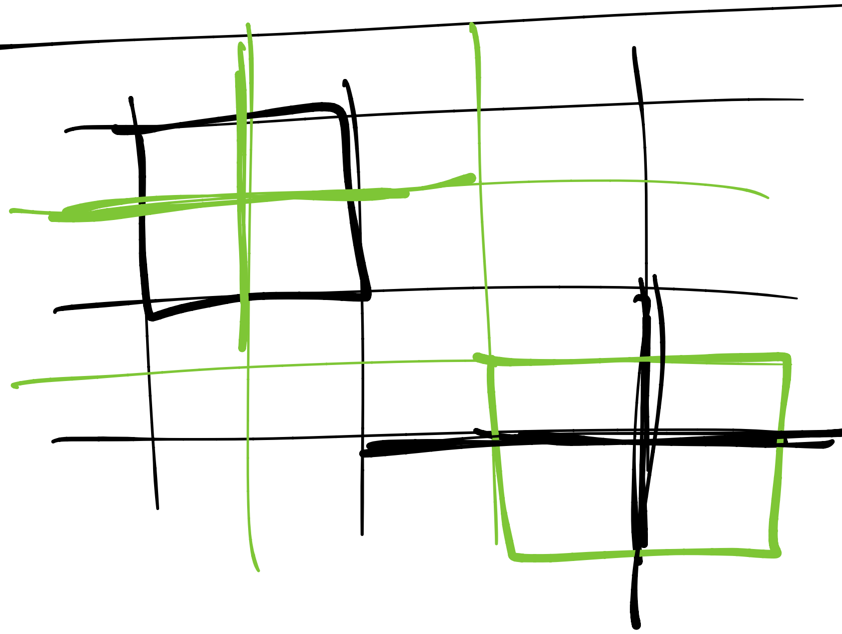
X basis: irreps of A.

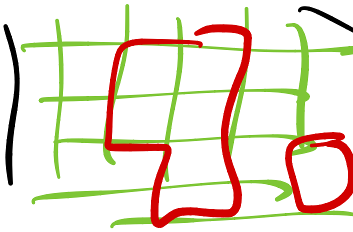
HINT for Non-Abelian case.

$$Z = \sum_{n=0}^{N-1} \omega^n |n\rangle\langle n|$$

$\in \mathbb{Z}_N$

$$X = \sum |n+1\rangle\langle n|$$



$$|gs\rangle = \sum_{\substack{\text{closed loops } c^\vee \\ \text{on dual} \\ \text{lattice}}} \Psi(c^\vee) \, \Big|$$  $$\Big\rangle$$

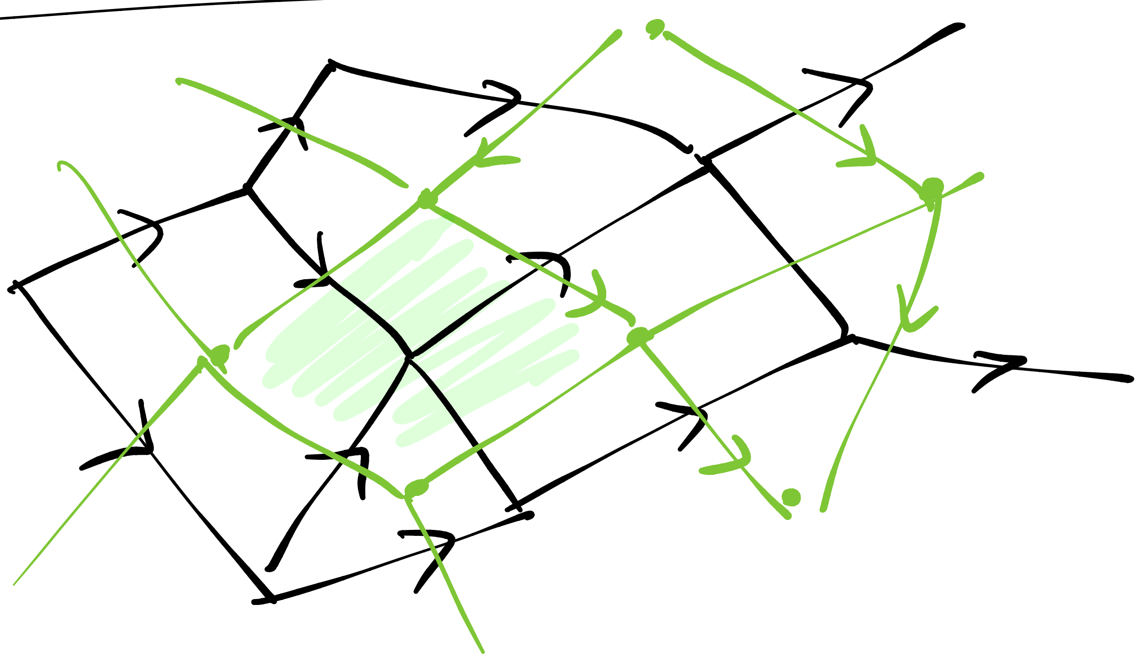
just as $|go'\rangle = W_c \sum_{\substack{\text{contractible} \\ \text{loops} \\ c}} |c\rangle$

$$W_c = \prod_{\ell \in c} X_\ell$$

$$|go''\rangle = V_{c^\vee} \sum_{\substack{\text{contractible} \\ \text{loops} \\ \text{on dual lattice}}} |c^\vee\rangle$$

$$V_{c^\vee} = \prod_{\ell \perp c^\vee} Z_\ell$$

$$\Delta_p = \Delta^\vee_{d-p}$$



dual bdy map : $\partial^\vee$

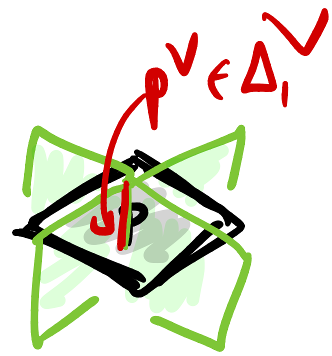
$$\boxed{\partial^\vee \underbrace{(\sigma_p)^\vee}_{\in \Delta^\vee_{d-p}} \equiv \underbrace{\big(\underbrace{V(\sigma_p)}_{\in \Delta_{p+1}}\big)^\vee}_{\in \Delta^\vee_{d-p-1}}} .$$

$\boxed{\text{Either } \Delta \text{ is oriented or take } A = \mathbb{Z}_N}$

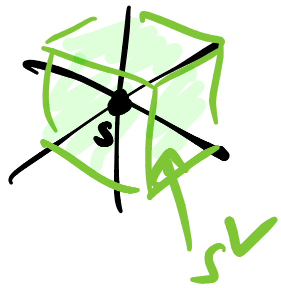
<u>3 d</u> : $\underbrace{\text{1-form toric code}}$ $\Longleftrightarrow$ 2-form toric code

write the $|gs\rangle$ in X-basis:

$$= \sum_{\substack{\text{closed} \\ \text{strings on } \Delta \\ c}} |c\rangle$$



$B_p = 1 \Longleftrightarrow$ membranes on dual lattice are closed at $p^\vee$.



$$A_s = \prod_{w \in \tilde{\partial}(s^\vee)} Z_w$$

star op. for 2-form Tr C.

$$|gs\rangle = \sum_{\substack{\text{closed} \\ \text{curves } C \\ \text{on } \Delta}} |C\rangle = \sum_{\substack{\text{closed} \\ \text{membranes} \\ M \text{ on } \Delta^\vee}} |M\rangle .$$

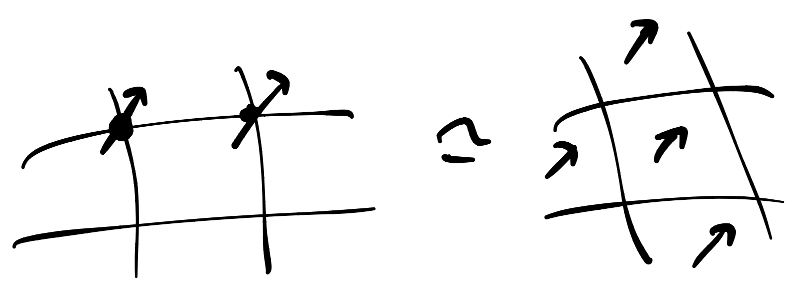
<u>Poincaré duality</u> : $H_p(X, A) \cong H_{d-p}(X, A)$

if $A = \mathbb{Z}_2$ or X is orientable.

In particular $b_p(X) = b_{d-p}(X)$.

<u>More eg:</u> Any d, $p=0$.

$\cong$ $p = d$.



$\boxed{d=4,\ p=2}$ $\simeq$ $d=4,\ p=2$. self-dual

$$\underline{\Delta_2 = \Delta_2^\vee}.$$

<u>No particle excitations</u> (string operators)

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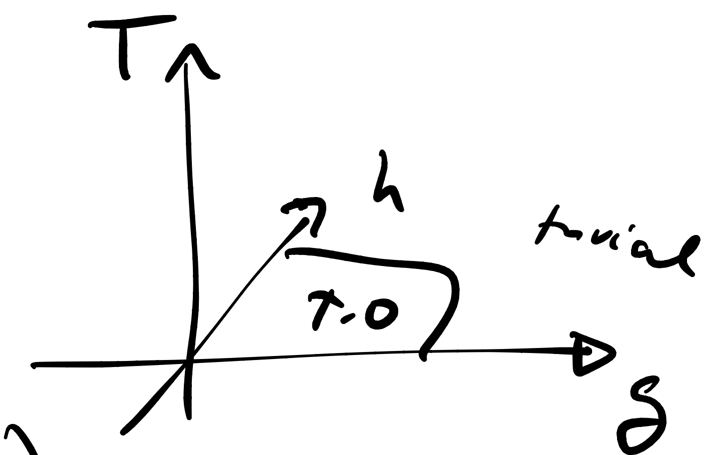
everything so far: $\underline{T=0}$.

$$|gs\rangle \underset{\xleftarrow{T\to 0}}{\rightsquigarrow} \rho = e^{-H/T}/Z$$

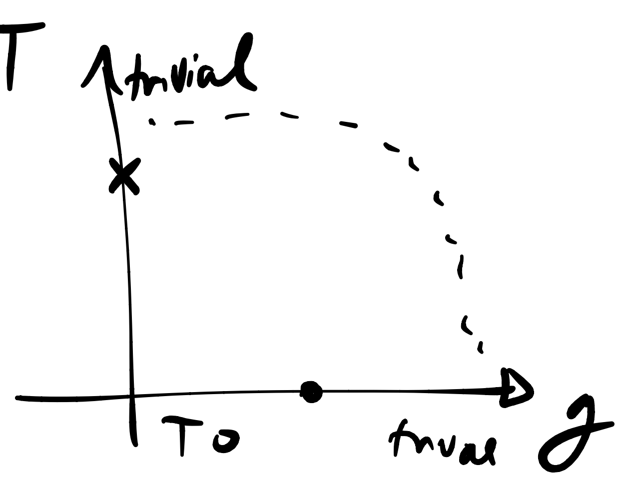
$$\Big\downarrow T\to\infty$$

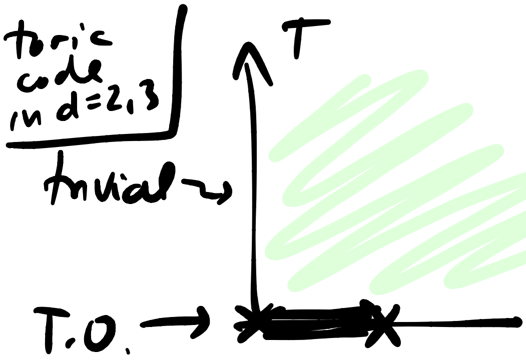
$$\rho_{T=\infty} = \frac{\mathbb{1}_{\mathcal{H}}}{\dim\mathcal{H}} = \bigotimes_{\ell}\left(\frac{\mathbb{1}_{\mathcal{H}_\ell}}{\dim\mathcal{H}_\ell}\right)$$

<u>PRODUCT STATE</u>, <u>trivial</u>

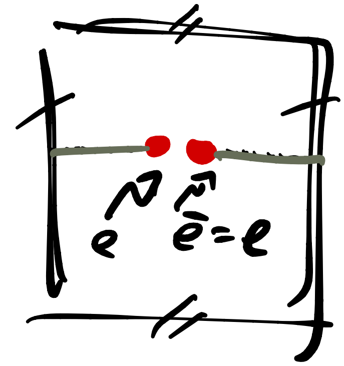


<u>claim</u>: In all cases so far, at any $T>0$, the state is trivial.





why:



$W_c \, |gs_{00}\rangle = |gs_{01}\rangle$

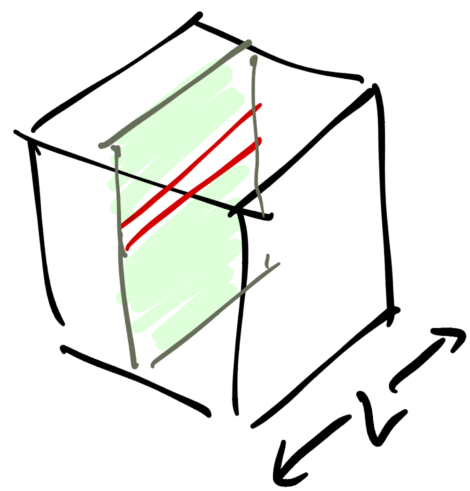
At $T>0$

$$n_e = n_{\bar{e}} \sim e^{-\# \Delta / T}$$

$\equiv$ density of e-particles

$\Delta \equiv$ energy gap $=$ energy to create an e-particle

In contrast: The $d=4$ $p=2$ TC has no particles!

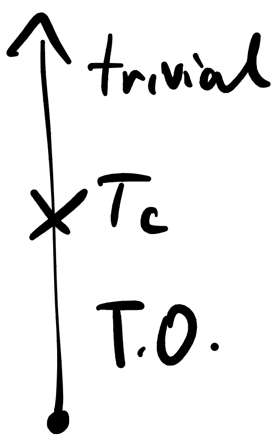


$$n_{\text{strings}} \sim e^{-\sigma L / T} \xrightarrow{L\to\infty} 0 \quad \text{at finite } T.$$

$TC_{\substack{d=4 \\ p=2}}$ is a finite-temp quantum memory!

Q: $\exists$ a finite temp quantum memory in $d=3$?

$d=4$
$p=2$

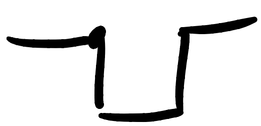


tension

Gas of tensionful strings : $E \sim \sigma L$

$\Omega(L) \equiv$ # of states of a string of length L $\sim \#^L$

$\Rightarrow S(L) \sim L$



2^{d-1} options at each step.

$$\Rightarrow \boxed{S(E) \sim E}$$

Hagedorn behavior.

$$F = E - TS$$
$$= \underbrace{(1 - aT)} L\sigma$$

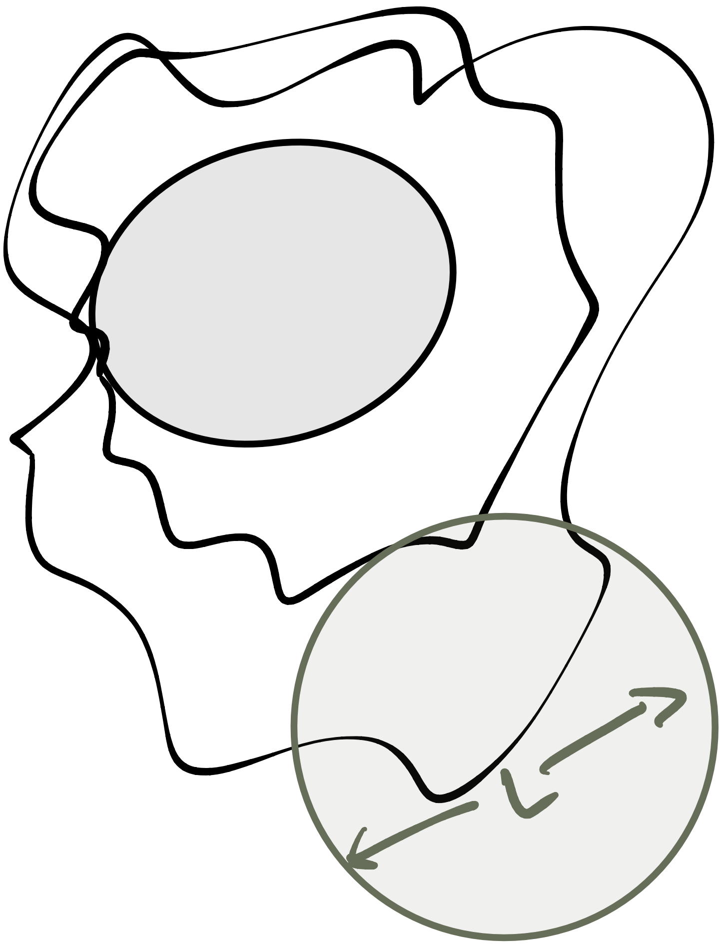
If $1-aT > 0 \to L \to 0$
$1-aT < 0 \to L \to \infty$

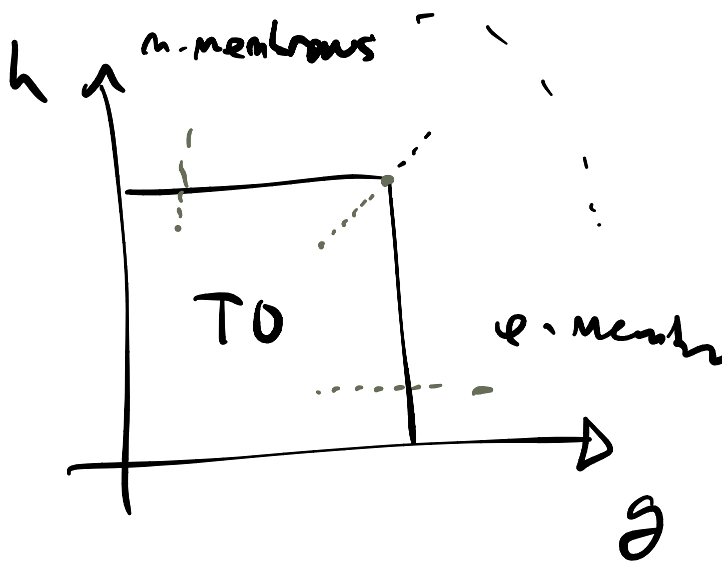
$\underline{T_c \sim 1/a}$.

$\underline{T<T_c}$: dilute gas of small strings (T.O.)

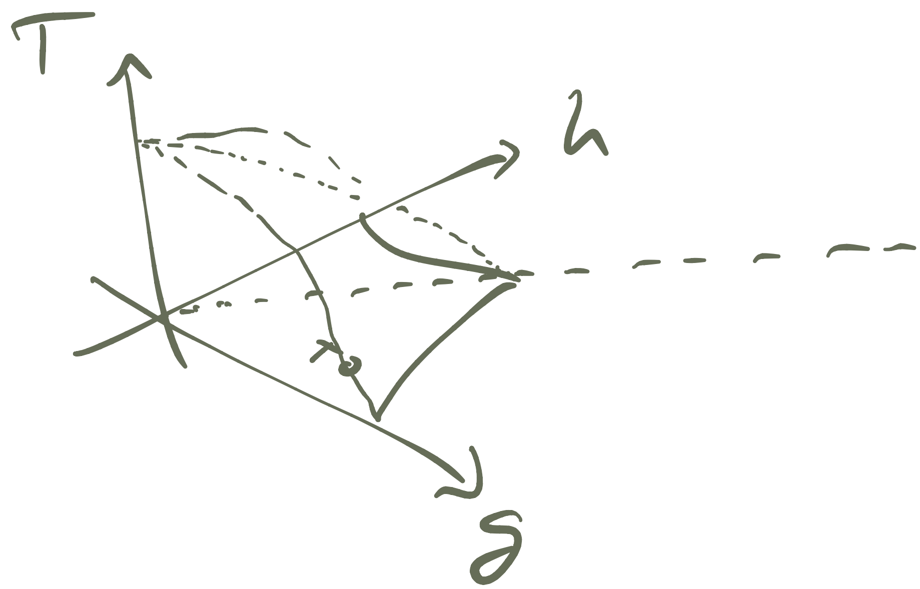
$\underline{T>T_c}$: big strings condense

~ trivial phase.





d=4
p=2

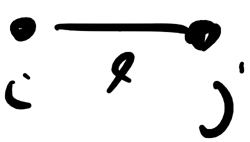


Discrete g.t. participate in another duality.

Kramers-Wannier-Wegner :

$$H_{clock} = -\Gamma \sum_{\substack{\ell\in\Delta_\ell \\ \partial\ell = i-\ell}} X_i X_j^\dagger - g \sum_{i\in\Delta_0} Z_i + h.c.$$

$\mathbb{Z}_N$ dofs on 0-cells of Δ



($N=2$: TFIM)

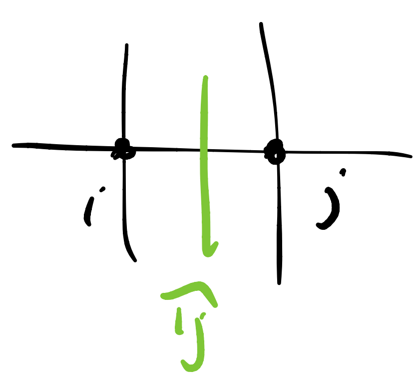
$$X_i = e^{2\pi i k_i/N}$$

$$X_i X_j^\dagger = -2\cos\frac{2\pi}{N}(k_i-k_j)$$

($p=0$ any d.)

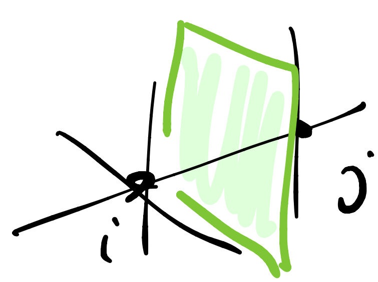
put a $\mathbb{Z}_N$ var on d-1 cells of $\Delta^\vee$.

$$\boxed{\sigma^z_w \equiv X_i X_j^\dagger}$$



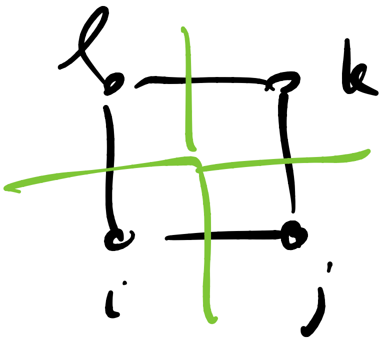
$w=(ij)^\vee$

Domain wall variable.



Not all configs of σ^z are allowed.

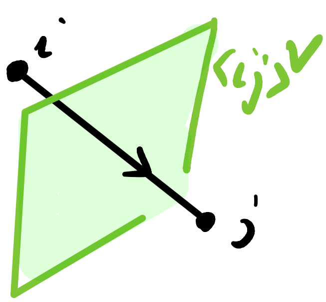
$$1 = \prod_{\ell \in \partial p} \sigma^z_\ell \qquad \forall p \in \Delta_2 = \Delta_{d-2}^\vee$$

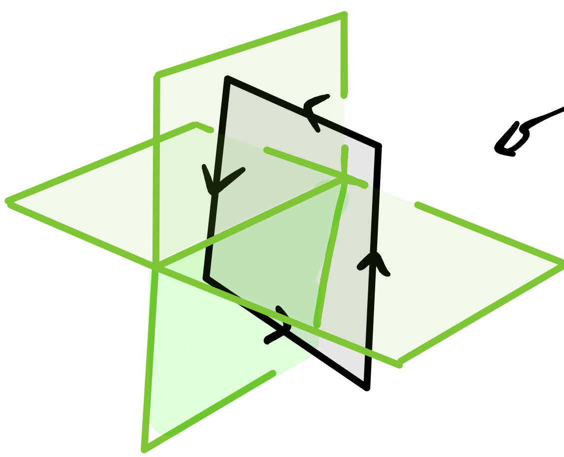


In $d=2$:

star term for 1-form T.C.

$\boxed{p=0,\ d=3}$



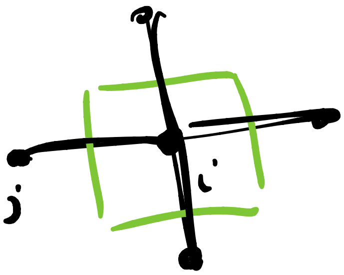


$\prod \sigma^z = 1$ gives star term for 2-form T.C.

$$TC\begin{pmatrix} p=0 \\ d \end{pmatrix} \longleftrightarrow TC\begin{pmatrix} d-p-1 \\ d \end{pmatrix}$$

w/ exact gauss law.

$$Z_i = \prod_{\ell \in v(i)} \sigma^x_{\ell \in \Delta^\vee_{d-1}} = \text{plaquette operator!}$$



$$(*\omega_q)_{i_{q+1}\cdots i_d} = \frac{\epsilon_{i_1\cdots i_d}\,\omega^{i_1\cdots i_q}\sqrt{g}}{q!}$$

is $d-q$ form

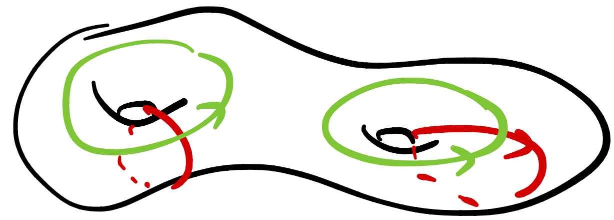
$$\boxed{\eta \wedge *\omega = (\eta,\omega)\,\mathrm{vol}.} \quad \forall \eta.$$

$$H_p(X) \simeq H^{d-p}_{\text{compact}}(X)$$

$$S = \int d\phi \wedge *d\phi + V(\phi)$$

$$d\phi_p = *_{d+1} \underbrace{d\tilde{\phi}_{d-p-1}}_{d-p}$$

CS on Σ_2



$\{|gs\rangle\} \sim \{$ conformal blocks of WZW CFT. $\}$

$$H_1(T^2, \mathbb{Z}_N) = \mathbb{Z}_N^2$$

$$\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right) \equiv I$$

vs: # of gs of $U(1)_m$ CS

$= m$.

abelian: $H_1(\Sigma, \mathbb{Z})$ has a intersection form

choose a maximal Lagrangian subspace of $H_1(\Sigma, \mathbb{Z})$

$$\mathcal{L} = \{ C \in H_1(\Sigma, \mathbb{Z}_m) \text{ s.t. } C_i \cap C_j = 0 \ \forall i, j \}$$

$$\begin{cases} W_C = e^{i \oint_C A} \\ W_{C_i} W_{C_j} = \omega^{I(i,j)} W_{C_j} W_{C_i} \end{cases}$$

$$\{g_L\} \leftrightarrow \mathbb{Z}_m^{\dim \mathcal{L}} = \mathcal{L}.$$