last time: susy QM algebra:

$[Q, H] = 0$. $\{Q, Q^\dagger\} = 2H$.

$\boxed{Q^2 = 0}$.

susy & cohomology: $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$

$(-1)^F = 1$ $(-1)^F = -1$

$\xrightarrow{Q} \mathcal{H}^B \xrightarrow{Q} \mathcal{H}^F \xrightarrow{Q} \mathcal{H}^B \to \cdots$

$\{(-1)^F, Q\} = 0$.

is a complex.

$$H^B(Q) = \frac{\ker Q : \mathcal{H}^B \to \mathcal{H}^F}{\operatorname{im} Q : \mathcal{H}^F \to \mathcal{H}^B} \cong \operatorname{span}\{\text{bosonic susy g.s}\}$$

$$H^F(Q) = \frac{\ker : Q : \mathcal{H}^F \to \mathcal{H}^B}{\operatorname{im} : Q : \mathcal{H}^B \to \mathcal{H}^F} \cong \operatorname{span}\{\text{fermionic susy g.s.}\}$$

Given $|\alpha\rangle$ w/ $H|\alpha\rangle = E|\alpha\rangle$.

suppose $|\alpha\rangle$ is Q-closed: $Q|\alpha\rangle = 0$.

claim: $|\alpha\rangle = Q|\beta\rangle$. $|\alpha\rangle$ is exact.

Pf: Act on $|\alpha\rangle$ by $1 = \frac{QQ^\dagger + Q^\dagger Q}{2E}$.

$$\Rightarrow |\alpha\rangle = Q\Big(\underbrace{\frac{Q^\dagger}{2E}|\alpha\rangle}_{= |\beta\rangle}\Big) \qquad \blacksquare$$

Important special case: suppose

$[F,H]=0$. $F \in \mathbb{Z}$, $[F,Q]=Q$.

such an F generates a $U(1)$ R-symm.

$$\cdots \to \mathcal{H}^{p-1} \xrightarrow{Q} \mathcal{H}^{p} \xrightarrow{Q} \mathcal{H}^{p+1} \xrightarrow{Q} \cdots$$

eigenspace of F w/ eval $p-1$

$$\mathbf{H}^p(Q) \equiv \frac{\ker Q : \mathcal{H}^p \to \mathcal{H}^{p+1}}{\operatorname{im} Q : \mathcal{H}^{p-1} \to \mathcal{H}^p}.$$

$$\operatorname{tr}(-1)^F = \underbrace{\sum_{p \in \mathbb{Z}} (-1)^p \dim \mathbf{H}^p(Q)}_{\text{euler character of } \mathcal{H}^\bullet}.$$

- cohomology: Q increases p. (reversal of arrows)
- given Q, $Q^2=0 \Rightarrow$ can construct $\mathbf{H}^\bullet(Q)$. expect: invariants

- Consider a susy QFT in D dims.

$$\{\underline{\underline{Q_\alpha}}, Q_\beta\} = 2\,\underline{\gamma^\mu_{\alpha\beta}} P_\mu + \ldots$$

Put the QFT on a mfld X.

$\longrightarrow$ inv'ts of X? $\left\{\begin{array}{l}\cdot\ \text{curvature of } X \text{ breaks SUSY} \\ \cdot\ \text{no global spinors on } X.\end{array}\right.$

topological twist : <u>change the spin of Q_α</u> (using R symmetry) to make Q a scalar, $Q^2 = 0$.

Cohomology of Q $\Longrightarrow$ <u>inv'ts of X.</u>

General algebraic fact:

<u>Note</u>: $K = \frac{Q^\dagger}{2E}$ satisfies $\underline{\underline{QK + KQ = 1}}$ (on excited state)

$K : \Omega^p \to \Omega^{p-1}$

$Q : \Omega^p \to \Omega^{p+1}$

$\left\{\begin{array}{l}\exists K \text{ s.t. } \ldots \\ \Rightarrow \text{coho of } Q \text{ is trivial.}\end{array}\right.$

K = "<u>homotopy</u> operator".

SUSY QM:

$$H = \frac{1}{2}\{Q, Q^\dagger\} \quad \text{w/} \quad Q = \psi^\dagger(p - iW'(x))$$

$$\left\{ \begin{array}{l} \{\psi, \psi^\dagger\} = 1 \\ [x, p] = i \end{array} \right.$$

$\updownarrow$ Legendre

$$S[x,\psi,\psi^\dagger] = \int dt \left(\frac{1}{2}\dot{x}^2 + i\psi^\dagger\dot{\psi} - \frac{1}{2}(W'(x))^2 + \frac{1}{2}W''(x)[\psi^\dagger, \psi] \right) \bigstar$$

claim:

is inv't under SUSY:

+ total deriv.

$\frac{1}{2}\frac{d}{dt}(\psi^\dagger\psi)$

$$\delta_\epsilon \mathcal{O} \equiv i[\epsilon Q - \epsilon^\dagger Q^\dagger, \mathcal{O}]$$

$$\bigstar \left\{ \begin{array}{l} \delta_\epsilon x = \epsilon\psi - \epsilon^\dagger\psi^\dagger \\ \delta_\epsilon \psi = -i\epsilon^\dagger(\dot{x} + iW'(x)) \\ \delta_\epsilon \psi^\dagger = i\epsilon(\dot{x} - iW'(x)) \end{array} \right.$$

Side Remark about Superspace:

$\bigstar \sim$ translation on superspace

$\equiv$ space w/ coords $(t, \theta, \theta^\dagger)$

grassmann.

$\theta^2 = 0$

$\{\theta^\dagger, \theta\} = 0$

superfield $\equiv$ f'n on superspace (real)

$$X(t,\theta,\theta^\dagger) \equiv x(t) + \theta\psi(t) - \theta^\dagger\psi^\dagger(t) + \theta\theta^\dagger F$$

$$S = \int dt\, d\theta\, d\theta^\dagger\, L(X(t,\theta,\theta^\dagger))$$

is automatically susy inv't.

$$\boxed{\int d\theta\, \theta = 1 \qquad \int d\theta = 0 .}$$

$$\int d\theta\, d\theta^\dagger \left(+ \dots \theta\theta^\dagger Z \right) = Z .$$

[action of susy on X is

$$Q = \partial_\theta + i\bar\theta \partial_t .$$

↑ translation.]

eg: $\begin{cases} L = \frac{1}{2} DX\, DX + W(X) \\ D \equiv \partial_\theta - i\bar\theta\partial_t \end{cases}$

$$\rightarrow \int d\theta\, d\theta^\dagger\, L(X) = \frac{1}{2}\dot{x}^2 + i\psi^\dagger\dot\psi + \frac{1}{2}W''(x)[\psi^\dagger,\psi] - W'F + \frac{1}{2}F^2 .$$

F is auxiliary: $0 = \frac{\delta S}{\delta F} = -W' + F$. $\rightarrow$ ★.

(Susy) Nonlinear σ-Model (NLSM):

A QFT of **Maps**: $\mathbb{R}^D \longrightarrow \mathcal{M}_n$

$x^\mu \longmapsto \phi^i(x)$

base space

target space

$\mu = 1 \dots D$. $i = 1 \dots n$

[eg: $\mathcal{M} = G/H$ arises as a description of Goldstones when breaking $G \to H$.

$$S[\phi, \psi, \psi^\dagger] = \int d^Dx \left[\frac{1}{2}\gamma_{ij}(\phi)\left(\partial_\mu\phi^i\partial_\nu\phi^j + \bar\psi^i i\gamma^\mu D_\mu \psi^j\right) + \frac{1}{8}R_{ijkl}(\phi)\bar\psi^i\psi^k\bar\psi^j\psi^l\right]$$

$\gamma_{ij} \equiv$ R'n metric on $\mathcal{M}$.

R_{ijkl}: curvature of γ^{ij}.

$\gamma^\mu_{\alpha\beta}$ = γ-matrices

$$\underbrace{D_\mu\psi^i \equiv \partial_\mu\psi^i + \Gamma^i_{jk}\partial_\mu\phi^j\psi^k}.$$

$\Leftrightarrow \psi^i$ is a tangent vector.

ψ^i are Majorana fermions (2 components)

$\bar{\psi} \equiv \psi^\dagger \gamma^0$.

<u>claim</u>: For $D \leq 3$ this is supersymmetric.

claim: $D > 3$ constraint M.

<u>Q</u>: <u>what is $\mathrm{tr}(-1)^F$</u>? How many susy g.s.?

<u>Method 1</u>: classically, any config $\phi(t)$ has $E = 0$.

($\phi = \phi(x)$ costs energy $\sim \frac{1}{V}$.)

$$\begin{cases} \phi(t,x) = \phi(t). \\ \psi(t,x) = \psi(t) \end{cases}$$

plug into action.

(= Dim'l reduction)

$$S[\phi,\psi,\psi^\dagger] = \frac{V}{2}\int dt \Big[\gamma_{ij}(\phi)\big(\dot\phi^i\dot\phi^j + \bar\psi^i i D_0 \psi^j\big) + \frac{1}{4} R_{ijkl}(\phi)\bar\psi^i\psi^k\bar\psi^j\psi^l \Big]$$

$$\equiv \frac{V}{2}\int dt \left(\int d\theta d\theta^\dagger \, \gamma_{ij}(\Phi) D\Phi^i D\Phi^j . \right).$$

$i = 1..n$ $\Phi^i = \Phi^i(t,\theta,\theta^\dagger) = \phi^i + \theta\psi^i - \theta^\dagger\psi^{i\dagger} + \theta\bar\theta F^i$.

= QM of a particle moving on $\mathcal{M}$.
+ fermions.

equally: gs wave f'n spreads over $\mathcal{M}$
to minimize p^2.

($\mathcal{N}=1$ SUSY in $D=3 \Rightarrow 2$ real supercharges).

Pick a basis for $\gamma^0 = \sigma^3$ where
the Majorana condition is $\Psi = \begin{pmatrix} \psi \\ \psi^+ \end{pmatrix}$.

$$= U \begin{pmatrix} \psi + \psi^+ \\ \frac{\psi - \psi^+}{i} \end{pmatrix}.$$

$$p_i = \frac{\partial L}{\partial \dot\phi^i} = -i D_t \phi^i$$

$$\{\psi^i, \psi^j\} = 0, \quad \{\psi^i, \psi^{j*}\} = \gamma^{ij}(\phi).$$

supercharges are:

$$Q = i \sum_i \psi^{*i} p_i \qquad Q^+ = -i \sum_i \psi^i p_i.$$

$Q^2 = (Q^\dagger)^2 = 0$. $[H, Q] = 0$.

$\{Q, Q^\dagger\} = 2H$. fixes additive normalization of H.

S is inv't under a $U(1)_R$ sym:

$$\begin{cases} \psi^i \to e^{-i\alpha}\psi^i \\ \psi^{i\dagger} \to e^{+i\alpha}\psi^{i\dagger} \end{cases}$$

generated by

$F = \gamma_{ij}(\phi)\psi^{\dagger i}\psi^j$.

satisfies $[H, F] = 0$

$$\begin{cases} [F, Q] = Q. \\ [F, Q^\dagger] = -Q^\dagger. \end{cases}$$

$F|0\rangle = 0$

w/ $\psi^i|0\rangle = 0$.

claim: evals of $F \in \mathbb{Z}$.

Note: $\gamma_{ij}(\phi)\gamma^{jk}(\phi) = \delta_i^{\ k}$.

$\{\psi^i, \psi^{j\dagger}\} = \gamma^{ij}$

$\delta_\alpha O = i[\alpha \hat{F}, O]$

$= i[\alpha\gamma_{ij}(\phi)\psi^{\dagger i}\psi^j, O]$.

eg: $O = \psi^i$: $\delta_\alpha \psi^i = i\alpha \underbrace{\gamma_{jk}\{\psi^{\dagger j}, \psi^i\}}_{=\delta^i_k}\psi^k = i\alpha\psi^k$.

Build $\mathcal{H}$. let $|0\rangle$ satisfy $\psi^i|0\rangle = 0$.

reference vacuum.

$$A(\phi)|0\rangle$$

$$A_i(\phi)\psi^{+i}|0\rangle$$

$$A_{ij}(\phi)\psi^{+i}\psi^{+j}|0\rangle \qquad A_{ij} = -A_{ji}.$$

$$\vdots$$

$$\text{span}\left\{\underline{A_{i_1\ldots i_p}}(\phi)\psi^{+i_1}\ldots\psi^{+i_p}|0\rangle\right\} \qquad \underline{A_{i_1\ldots i_p}} \text{ is A.S.}$$

$$\equiv \Omega^p(M).$$

$$\vdots$$

$$A_{i_1\ldots i_n}(\phi)\underbrace{\psi^{+i_1}\ldots\psi^{+i_n}|0\rangle}_{=|\text{plenum}\rangle}. \qquad n \equiv \dim M.$$

eigenspace of F w eval p.

$$\Omega^p(M) \equiv \text{differential } p\text{-forms on } M.$$

$$A \equiv A_{i_1\ldots i_p}(\phi)\, d\phi^{i_1}\wedge\ldots\wedge d\phi^{i_p}.$$

$$\psi^{+i} = d\phi^i \qquad d\phi^i \wedge d\phi^j = -d\phi^j \wedge d\phi^i$$

under coord. change $y^I \longrightarrow \phi^i(y)$

$$d\phi^i = \frac{\partial \phi^i}{\partial y^I} dy^I .$$

$$\mathcal{H} = \bigoplus_{p=0}^{n} \Omega^p(M) .$$

de Rham <u>complex</u> on M.

<u>How does Q act?</u> $Q = \psi^{+i} p_i$

$p_i = -i D_{\phi_i}$, $\psi^{i+} = d\phi^i \wedge \ldots$

$\psi^i = \gamma^{ij} \, i_{\partial_{\phi^j}}$

$i_V : \Omega^p \to \Omega^{p-1}$

eg $A = A_{ijk} \, \underline{dx^i} \wedge dx^j \wedge dx^k$

$i_V A = V^i A_{ijk} \, dx^j \wedge dx^k .$

$$Q\,|A_q\rangle = Q\left(A_{i_1\cdots i_q}(\phi)\,\psi^{*i_1}\cdots\psi^{*i_q}|0\rangle\right)$$

$$= \underline{D_{\phi^j}} A_{i_1\cdots i_q}(\phi)\;\psi^{+j}\psi^{+i_1}\cdots\psi^{+i_q}|0\rangle$$

$$= \frac{1}{(q+1)!}\left(\underline{\partial_{\phi^{i_1}}} A_{i_2\cdots i_{q+1}}(\phi) \pm \text{perms}\right)$$

$$\psi^{+i_1}\cdots\psi^{+i_{q+1}}|0\rangle$$

$$\equiv |dA_q\rangle$$

$d: \Omega^q \to \Omega^{q+1}$ exterior derivative.

$$D_\mu A_{\cdot\nu\cdot} = \partial_\mu A_{\cdot\nu\cdot} \pm \underbrace{\Gamma^{\,i}_{\mu\nu}}_{\text{symmetric}} A_{\cdots i\cdots} + \cdots$$

$$H = \{Q^+, Q\}/2 \;\xrightarrow{Q^2=0}\; [Q, H] = 0$$

$$\int d\psi_1 \ldots d\psi_{2n} \, e^{\psi_i A_{ij} \psi_j} = \mathrm{Pf}(A).$$

$$\mathrm{tr}\,(-1)^F e^{-\beta H} \mathcal{O} = \int_{\psi(\tau) = +\psi(\tau+\beta)} D\phi D\psi \, e^{-S} \mathcal{O}$$

"BPS quantity"

$$= \sum_{\phi_* \text{ s.t.}} e^{-S[\phi_*]} \mathcal{O}[\phi_*]$$

$$\delta\psi = F[\phi]\Big|_{\phi = \phi_*} = 0.$$

$\underline{\mathcal{N} = 1}$.

$$H = P_0.$$

Lorentz $\Rightarrow$

$$\{Q_\alpha, Q_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu$$

$$F^\mu_{\alpha\beta} \gamma^\nu_{\alpha\beta} = \eta^{\mu\nu}$$

$$\Rightarrow H = \frac{1}{2} F_{\alpha\beta} \{Q_\alpha, Q_\beta\}$$

$\underline{\mathcal{N} > 1}$: $\{Q^I_\alpha, Q^J_\beta\} = \delta^{IJ} \, 2\gamma^\mu_{\alpha\beta} P_\mu + Z^{IJ}_{\alpha\beta}$