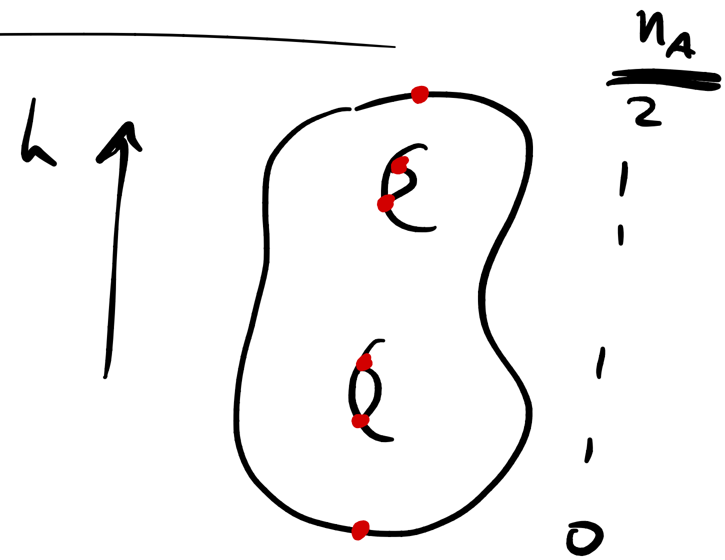
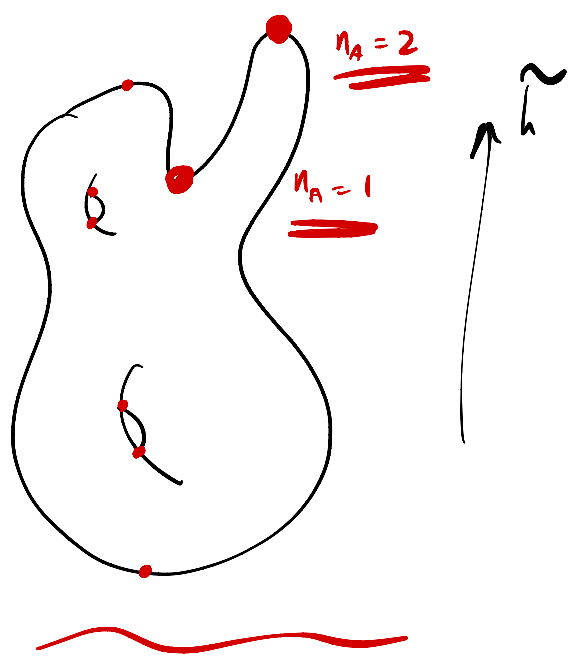
Morse thy & SUSY NLSM:

$$V = \frac{1}{2}|\partial h|^2$$

$$m_{ij} = \partial_i \partial_j h$$





Morse theory:

$b_q \le$ # of critical pts w/ Morse index q.

Tunneling: [Uses of Instantons, S. Coleman]

w/o fermions:

$$\langle B|\mathcal{O}\, e^{-HT}|A\rangle = \int_{\substack{\phi(-T/2)=\phi_A \\ \phi(T/2)=\phi_B}} \mathcal{D}\phi\; e^{-S_{eucl}[\phi]}\, \mathcal{O}[\phi]$$

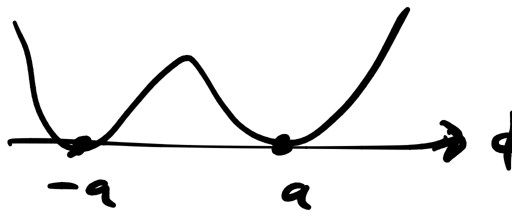
$$\simeq \sum_{\underline{\phi}} e^{-\underbrace{S_{eucl}[\underline{\phi}]}_{=S_0}} \mathcal{O}[\underline{\phi}] \det{}^{-1/2}\left(\frac{\delta^2 S}{\delta \phi^2}\Big|_{\underline{\phi}} \right)$$

"instantons"

$$\frac{\delta S_{eucl}}{\delta \phi}\Big|_{\phi = \underline{\phi}} = 0$$

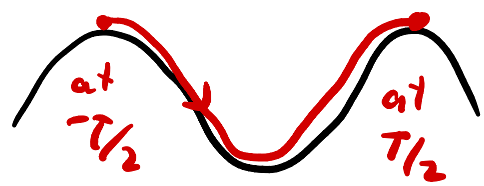
w/ B.C.

eg:



$\phi \to -\phi$ sym is restored by instantons

$\underline{\phi}$ solves eom for



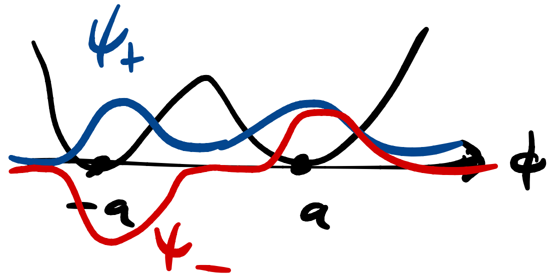
$$\langle \pm a | e^{-HT/\hbar} | a \rangle = \sqrt{\frac{\omega}{\hbar\pi}}\, e^{-\omega T/2} \sum_{\substack{\text{even/odd} \\ n}} \frac{\left(KT e^{-S_0/\hbar}\right)^n}{n!} \left(1 + \mathcal{O}(\hbar)\right)$$

$$\omega = V''(\pm a)$$

$$= \sqrt{\frac{\omega}{\hbar\pi}}\, \frac{1}{2}\left(e^{KT e^{-S_0/\hbar}} \mp e^{-KT e^{-S_0/\hbar}} \right)$$

$$\Rightarrow E_\pm = \frac{1}{2}\hbar\omega \mp \hbar K e^{-S_0/\hbar}$$

true min:



For NLSM:

$\partial_i \equiv \frac{\partial}{\partial \phi^i}$

$$S_{\text{eucl},B}[\phi] = \frac{1}{2}\int d\tau \left(\gamma_{ij}\,\partial_\tau\phi^i\,\partial_\tau\phi^j + t^2\gamma^{ij}\,\partial_i h\,\partial_j h\right)$$

$$= \frac{1}{2}\int d\tau\, \underbrace{\left|\partial_\tau\phi^i \pm t\,\gamma^{ij}\partial_j h\right|^2}_{\geq 0} \mp t\int d\tau\,\partial_\tau h$$

$\left(|v|^2 \equiv \gamma_{ij}v^i v^j\right)$

$$\geq t\,\left|h(\tau=\infty) - h(\tau=-\infty)\right|$$

$$= t|\Delta h|.$$

equality $\Longleftrightarrow$ $0 = \partial_\tau\phi^i \pm t\,\gamma^{ij}\,\partial_{\phi^j} h$

gradient flow by height fn!

$$S[\underline{\phi}^{A\to B}] = t\,|\Delta h|.$$

w/ ferms:

$$\langle B|\mathcal{O}\, e^{-HT}|A\rangle = \int_{BC} D\phi\, D\psi\, D\bar\psi\; e^{-S_{cl}[\phi,\psi,\bar\psi]}\,\mathcal{O}\dots$$

$$\simeq \sum_{\underline{\phi},\,\psi=0} e^{-S_{eucl,B}[\underline{\phi}]}\, \det{}^{-1/2}(\underline{\underline{B}})\; \underbrace{\int D\psi\, D\bar\psi\; e^{\int \bar\psi\, \underline{\underline{F}}\, \psi}}_{=\det F}\; \mathcal{O}(\phi)$$

claim: SUSY $\Rightarrow$ $B = F^\dagger F$.

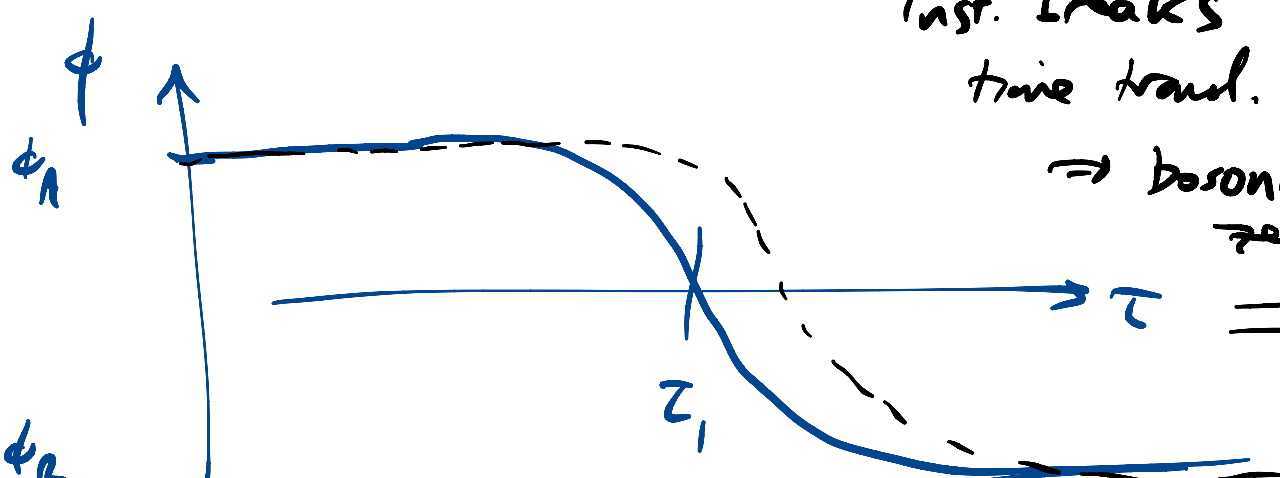
If $B\xi = \lambda\xi$ then let $\eta \equiv F^\dagger \xi/\sqrt{\lambda}$

then $\begin{cases} F\eta = \sqrt{\lambda}\,\xi \\ F^\dagger \xi = \sqrt{\lambda}\,\eta \end{cases}$

$$\overset{?}{\sim} \sum_{\underline{\phi},\,\psi} e^{-S_{eucl,B}[\underline{\phi}]}\, \mathcal{O}(\underline{\phi}).$$

inst. breaks time transl. inv

$\Rightarrow$ bosonic zero mode.



inst. breaks susy $\Rightarrow$ fermionic z.m.

$$\int d\tau_1 \int d\bar\eta \; (\underbrace{\ldots}_{\text{ind. of } \bar\eta}) = 0$$

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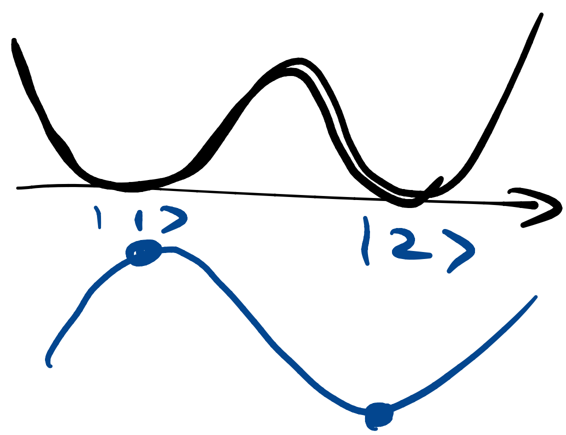
simple eg: $M = \mathbb{R}$, $h(x) = g x^3 - x$



Note: $\langle 1 | H | 2 \rangle = 0$

since $\begin{cases} (-1)^F |2\rangle = |2\rangle \\ (-1)^F |1\rangle = -|1\rangle . \end{cases}$

Final answer: $E_0 = \langle 0_+ | H | 0_+ \rangle$ ← true g.s.

$= \frac{1}{2} \langle 0_+ | \{ Q, Q^\dagger \} | 0_+ \rangle$ (suppose wlog $Q|0_+\rangle = 0$)

$= \frac{1}{2} \langle 0_+ | Q \, \mathbb{1} \, Q^\dagger | 0_+ \rangle \geq \frac{1}{2} |\underbrace{\langle 0_- | Q^\dagger | 0_+ \rangle}_{\equiv \epsilon}|^2$

$\mathbb{1} = \sum_{\pm} |0_\pm\rangle\langle 0_\pm| + \ldots$

SUSY $\Rightarrow$ $Q|0_+\rangle = |0_-\rangle$. are degenerate.

$$\epsilon = \langle 0_- | Q^+ | 0_+ \rangle \propto$$

$$\langle 1 | 0_- \rangle \langle 0_- | Q^+ | 0_+ \rangle \langle 0_+ | 2 \rangle$$

$$= \lim_{T \to \infty} e^{+E_0 T/\hbar} \langle 1 | e^{-H(T/2 - t_0)} Q^+ e^{-H(T/2 + t_0)} | 2 \rangle$$

$$= \int_{x(-\infty) = x_1}^{x(+\infty) = x_2} Dx \, D\psi \, D\bar{\psi} \, e^{-S_{eucl}/\hbar} Q^+(t_0)$$

$\uparrow$ $\langle x | 1 \rangle \sim \delta(x - x_1)$.

$$\mathcal{L}_E = \frac{1}{2} \dot{x}^2 + \frac{1}{2} W'^2 - \bar{\psi}\dot{\psi} - W'' \bar{\psi} \psi$$

<u>eom</u>: $\ddot{x} - W' W'' = 0$ w/ b.c.

cons. of "energy": $0 = \dot{x}^2 - W'^2 \Rightarrow \dot{x} = \pm W'$ (gradient flow)

$$S_0 = \Delta W = W(x_2) - W(x_1).$$

<u>fluctuations</u>: $S[x = \underline{x} + \delta x, \psi = \delta\psi, \bar{\psi} = \delta\bar{\psi}]$

$$= \Delta W + \frac{1}{2} \int d\tau \left(\delta x \, B \, \delta x + \delta\bar{\psi} \, F \, \delta\psi \right) + O(\delta^3)$$

w/ $F = \partial_t + W''$

$$B = -\partial_t^2 + W''' W' + (W'')^2 = \underbrace{(-\partial_t - W'')}_{F^+} \underbrace{(\partial_\tau - W'')}_{F}$$

$$B = F^{+} F$$

if no zms, $\frac{\det F}{\det^{1/2} B} = 1.$

~~X~~

$\Rightarrow \; \epsilon = e^{-\Delta W} \left(Q^{+}(t_0)\big|_{cl} + b(t_1) \right)$

BAD: – depends on t_0

– a grassman variable.

zeromodes: $\delta x = \delta\tau \, \dot{\underline{x}}$ $\Big\}$ has

$F \dot{\underline{x}} = (\partial_\tau - W'') \dot{\underline{x}}$

and $\delta \bar{\psi} = \bar{\eta} \, \dot{\underline{x}}.$

$= (\partial_\tau - W'') W'$

(no ψ z.m.)

$= W'' W' - W'' W'$

$= 0.$

$\underline{X}(t, \theta, \bar{\theta}) = \underline{x}(t) - \bar{\theta} \bar{\eta} \, \dot{\underline{x}}(t)$

satisfies $[H, \underline{X}] \neq 0, \; [Q^{+}, \underline{X}] \neq 0$

but $[Q, \underline{X}] = 0$

^ "collective coords of instanton"

integrate over

$$\epsilon \simeq \int d\bar{\eta} \int_{-T/2}^{T/2} dt_1 \; e^{-\Delta W/\hbar} \left(\bar{ce}(t_0)\Big| + O(\hbar) \right)$$

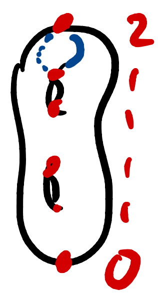
$$\bar{\psi}(\dot{x} - W')\Big|$$

$$= e^{-\Delta W/\hbar} \left(\underbrace{\int d\bar{\eta} \int dt_1 \, \bar{\eta}}_{=1} \; \dot{x}(t_1 - t_0)(\dot{x} - W') + O(\hbar) \right)$$

$$= e^{-\Delta W/\hbar} \left(\int_{x_1}^{x_2} dx \, (W' - W') + O(\hbar) \right)$$

$$= e^{-\Delta W/\hbar} \left(0 + O(\hbar) \right).$$

$$\langle B| \, Q_t \, e^{-HT} |A\rangle = \begin{cases} \pm \, e^{-t\Delta h/\hbar} & \text{IF } \; n_B - n_A = 1. \\ 0 & \text{else.} \end{cases}$$



$$\delta |A\rangle = \sum_{\substack{B \text{ with} \\ n_B - n_A = 1}} e^{-t(h(B)-h(A))} \, n(A,B) |B\rangle$$

$$n(A,B) = \sum_{\substack{\text{instantons} \\ \text{from } A \text{ to } B}} (\pm 1)$$

↑ compare orientations.

Floer homology: $\mathcal{M} \equiv$ space of G-gauge fields on X_3

A 1-form on $\mathcal{M}$ is $\delta A_a(x) = \psi_a(x)$

ext. deriv is field variation.

i.e. G valued fermion field.

Morse f'n, h : $S_{CS}[A) = \frac{k}{4\pi} \int_M tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$

$Q = d \to e^{-h} d e^{+h}$

$H = \{Q, Q^{\dagger}\}/2$.

$\longrightarrow$ susy g.s. are 3-mfld inv'ts

$\longrightarrow$ related to Donaldson thy of 4-mflds on $\mathbb{R} \times X_3$.

2.4 Global info. from local info

why cohomology?

A smooth map $f : M \to N$ we get a map

on $f^* : \Omega^0(N) \to \Omega^0(M)$.

by $f^*(g) \equiv g \circ f$. "pullback"

i.e. $g : N \to \mathbb{R}$, $f^*(g) : M \to \mathbb{R}$ by $f^*(g)(m) = g \circ f(m)$.

$\rightarrow f^*: \underline{\underline{\Omega^p(N)}} \to \Omega^p(M)$

$$f^*(\underbrace{g_{i_1\cdots i_p}\, dy^{i_1}\wedge\cdots\wedge dy^{i_p}}_{p\text{-form on } N}) = g_{i_1\cdots i_p}\circ f \; \underline{\underline{df_{i_1}\wedge\cdots\wedge df_{i_p}}}$$

y^i coords on N

$y^i : \text{nbhd } N \to \mathbb{R}$

$f_i \equiv y_i \circ f$

$= f^*(y_i)$

are local coords on M.

$f_i : \text{nbhd of } M \to \mathbb{R}$.

__claim:__ f^* is a chain map.

i.e. $[f^*, d] = 0$.

(chain rule)

$\underline{\underline{\Omega^\bullet}}$: smooth manifolds & smooth maps $\longrightarrow$ commutative graded differential algebra & homomorphisms.

$\uparrow$ a category $\qquad\qquad\qquad \uparrow$ a category.

is a covariant <u>functor</u>

(i.e. reverses arrows)

Poincaré lemma:

cohomology of $\mathbb{R}$ (warmup)

$\Omega^0(\mathbb{R}) \oplus \Omega^1(\mathbb{R}) \ni f_0(t) + f_1(t)\,dt$

is closed if $f_0' = 0$.

$\Longrightarrow H^0(\mathbb{R}) = \text{span}\{\text{constant}\} = \mathbb{R}$

But $d\left(\int_0^t f_1(t')\,dt' \right) = f_1(t)\,dt$

is exact.

$\longrightarrow H^1(\mathbb{R}) = 0.$

$$H^q(\mathbb{R}^n) = \delta^{q,0}\,\mathbb{R} = H^q(pt)$$

<u>claim</u>: $H^\bullet(M \times \mathbb{R}) \cong H^\bullet(M)$.

$$\begin{tikzcd} M \times \mathbb{R} \arrow[d, shift left, "\pi"] \\ M \arrow[u, shift left, "s"] \end{tikzcd} \qquad \begin{tikzcd} \Omega^\bullet(M \times \mathbb{R}) \arrow[d, shift right, "s^*"'] \\ \Omega^\bullet(M) \arrow[u, shift right, "\pi^*"'] \end{tikzcd}$$

$\pi(x,t) = x$ proj.

$s(x) = (x,0)$

"zero section"

$\pi \circ s = 1. \Rightarrow s^* \circ \pi^* = 1.$

But $s \circ \pi \neq 1 \Rightarrow \pi^* \circ s^* \neq 1$ on $\Omega^\bullet$

But claim: $\pi^* \circ s^* = 1$ on $H^\bullet(M \times \mathbb{R}) \overset{\pi^*}{\cong} H^\bullet(M)$.

CLAIM:

Pf: $\boxed{\underbrace{1 - \pi^* \circ s^*}\ = (-1)^{q-1}(dK - Kd)}$

ie $K : \Omega^q(M \times \mathbb{R}) \to \Omega^{q-1}(M \times \mathbb{R})$

is a homotopy operator on $\mathrm{Im}(1 - \pi^* \circ s^*)$.

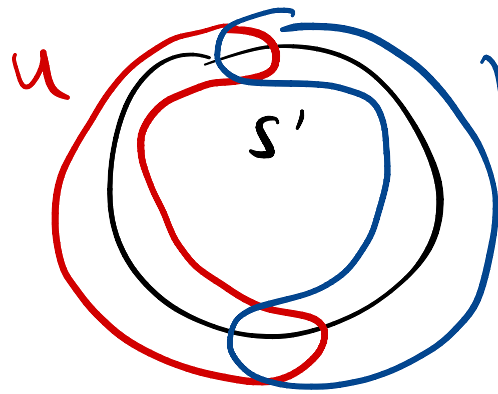
Any form on $M \times \mathbb{R}$ is $\qquad \phi_{0,1} \in \Omega^\bullet(M)$

$$(\pi^* \phi_0) \cdot f_0(x,t) + (\pi^* \phi_1) \cdot f_1(x,t)\, dt$$

$$\xrightarrow{K} 0 + (\pi^* \phi_1) \int_0^t f_1(x,t')\, dt' \quad \blacksquare$$

Mayer-Vietoris idea.

suppose $M = U \cup V$ U, V are open sets



$i_U : U \hookrightarrow M$

$i_V : V \hookrightarrow M$

$$0 \leftarrow U \cup V \xleftarrow{i} U \sqcup V \underset{i_V}{\overset{i_U}{\leftleftarrows}} U \cap V \leftarrow 0$$

disjoint union :

$$U \sqcup V \equiv \{(0,u) \mid u \in U\} \cup \{(1,v) \mid v \in V\}.$$

$\Longrightarrow$

$$0 \to \Omega^\bullet(U \cup V) \xrightarrow{i^*} \Omega^\bullet(U) \oplus \Omega^\bullet(V) \xrightarrow{i_U^* - i_V^*} \Omega^\bullet(U \cap V) \to 0$$

is **exact**.

$i_U : U \to M$ inclusion

$i_U^* : \Omega^\bullet(M) \to \Omega^\bullet(U)$ restriction.

$g \mapsto i^*(g) = g \circ i$

Pf: given $\omega \in \Omega^q(U \cap V)$

to show: $\omega = u - v$

w/ $u \in \Omega^q(U)$

$v \in \Omega^q(V)$.

partition of unity: $\equiv \{\rho_\alpha\}_{\alpha \in I}$ smooth fns

w. $\sum_\alpha \rho_\alpha = 1$ everywhere.

- every pt has a nbhd w/ $\sum_\alpha \rho_\alpha$ is a finite sum.

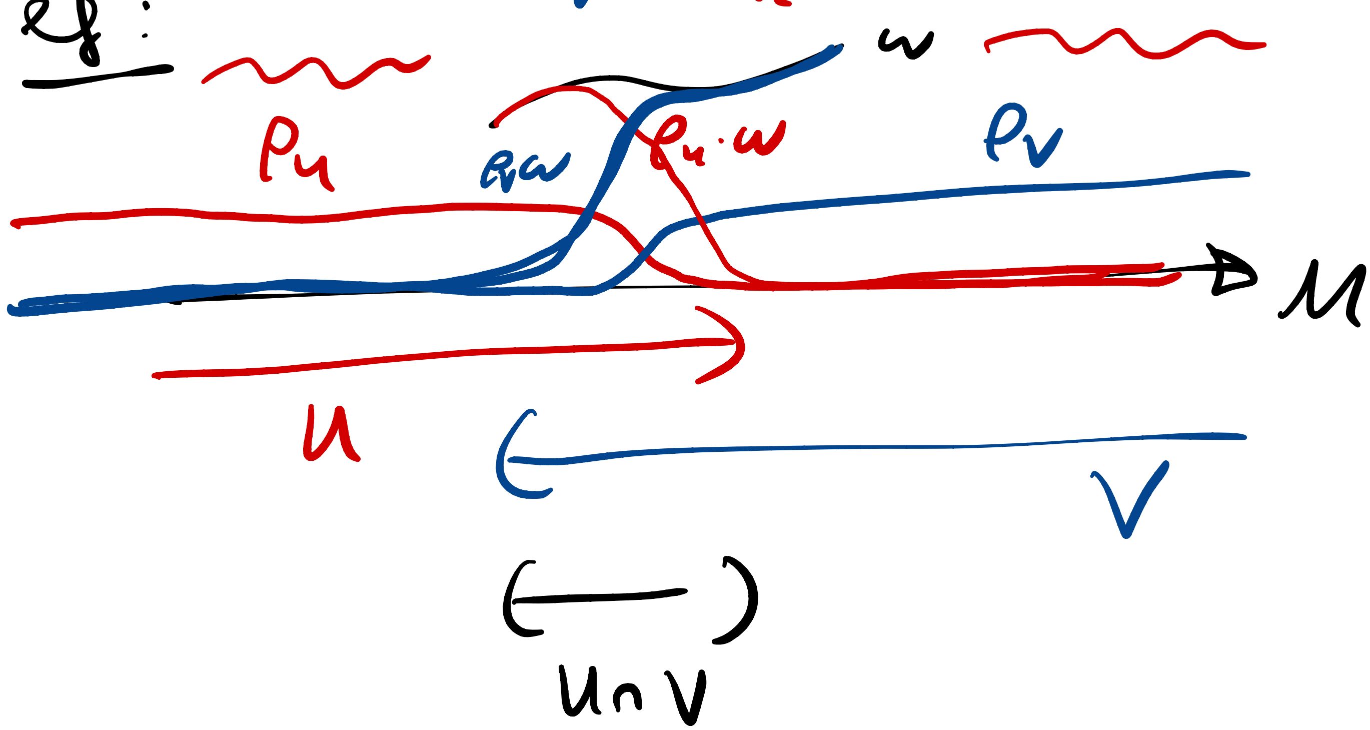
partition of unity subordinate to an open cover

$\mathcal{U} \equiv \{U_\alpha\}_{\alpha \in I}$ open sets w/ $\bigcup_\alpha U_\alpha = M$.

means (support of ρ_α) $\subset U_\alpha$.

$\rho_V + \rho_U = 1$ at each pt.

eg:



Back to $M = U \cup V$. $\quad \rho_U + \rho_V = 1$

$$\omega = \rho_U\,\omega + \rho_V\,\omega \qquad (\omega \in \Omega^\bullet(U\cap V))$$

$$= \underbrace{\rho_U\,\omega}_{\in\,\Omega^\bullet(V)} - (\underbrace{-\rho_V\,\omega}_{\in\,\Omega^\bullet(U)})$$

$\rho_U\,\omega = 0$ on $V \setminus (V\cap U)$

$\Longrightarrow$ long exact seq on cohomology:

$$\xrightarrow{d^*} \cdots$$

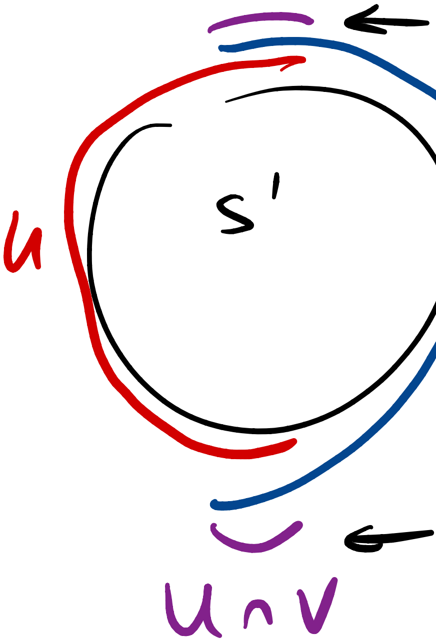
$$\xrightarrow{d^*} H^{q+1}(U\cup V) \longrightarrow H^{q+1}(U)\oplus H^{q+1}(V) \to H^{q+1}(U\cap V)$$

$$\to H^{q}(U\cup V) \longrightarrow H^{q}(U)\oplus H^{q}(V) \to H^{q}(U\cap V)$$

$$\cdots$$

given $[\omega] \in H^q(U\cap V)$

$$d^*[\omega] = \left\{ \begin{array}{ll} [-d(\rho_V\,\omega)] & \text{on } U \\ [d(\rho_U\,\omega)] & \text{on } V \end{array} \right\} \in H^{q+1}(U\cup V)$$



$$S^1 = U \cup V \qquad U \amalg V \qquad U \cap V$$

$$\begin{tikzcd}
H^1(S^1) & ? \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, "d^*"'] \\
H^0(S^1) & ? \arrow[r] & \underline{R \oplus R} \arrow[r, "\delta"] & \underline{R \oplus R} \\
& 0 \arrow[u]
\end{tikzcd}$$

$$\Longrightarrow H^0(S^1) = \ker \delta$$

$$H^1(S^1) = \operatorname{coker} \delta$$

$$H^q(\text{ball}, R) = R\,\delta^{q,0}$$

$$\uparrow$$

$$R^n$$

(to be shown)

$$\delta(\omega, \tau) = (\tau - \omega, \tau - \omega)$$

has has rank 1

$$\Longrightarrow H^0(S^1; R) \cong H^1(S^1; R)$$

$$\cong R.$$

<u>Note</u>: same as $H_0(S^1, R)$ and $H_1(S^1, R)$.

coincidence?!?