Mayer-Vietoris Idea: $H^\bullet(U), H^\bullet(V), H^\bullet(U\cap V)$ + $\underbrace{\text{gluing data}}_{\text{local info}} \to \underbrace{H^\bullet(U\cup V)}_{\text{global info}}$

Goals for today:

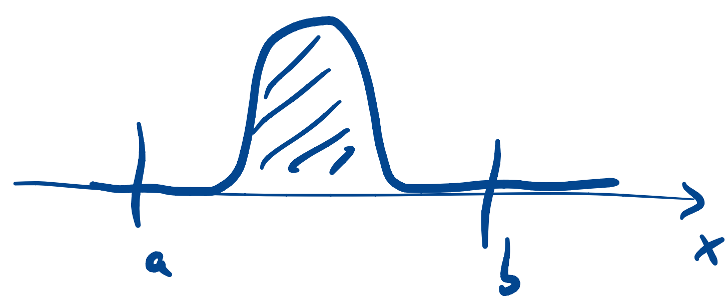
① Rel'ns betw. cohomology & homology

② Čech cohomology

Compactly supported Cohomology.

$$\Omega^\bullet_c(M) = \{\text{forms on } M \text{ w/ compact support}\}$$

$$d : \Omega^p_c(M) \to \Omega^{p+1}_c(M)$$



$$\to H^\bullet_c(M)$$

<u>eg:</u> $H^0_c(\mathbb{R})$. $df = 0$ requires $f'(x) = 0$

$\xRightarrow{f \in \Omega^0_c} f = 0$. $\quad \underline{H^0_c(\mathbb{R}) = 0}$.

If support$(f) \subset (a,b)$

$$\int_{\mathbb{R}} df = \int_{-\infty}^{\infty} dx\, f' = \int_a^b dx\, f'(x) \overset{FTC}{=} f(b) - f(a) = 0$$

$\in \Omega^1_c(\mathbb{R})$

ω is exact $\iff \underbrace{\int_{\mathbb{R}} \omega = 0.}$

$\omega = df$

($\boxed{\Leftarrow}$: $f(x) = \int_{-\infty}^{x} \omega \in \Omega^0_c(M)$

and $df = \omega$.)

$$H^1_c(\mathbb{R}) = \Omega^1_c(\mathbb{R}) \big/ \ker\left(\int\right) = \mathbb{R}.$$

Poincaré lemma for $H^q_c(\mathbb{R}^n) = \delta^{q,n}\, \mathbb{R}$

Moreover $H^{q+1}_c(M \times \mathbb{R}) \cong H^q_c(M)$

$\Omega^\bullet$: manifolds $\longrightarrow$ graded algebras

a contravariant functor.

using $f : M \to N$

$f_* : \Omega^\bullet(N) \to \Omega^\bullet(M)$

$\Omega^\bullet_c$ is not a functor under pullback.

eg: $\pi : M \times \mathbb{R} \to M$

$\pi_*(\omega) = \omega \circ \pi$ does not have compact support.

$\omega \in \Omega^\bullet_c(M)$

It is a Covariant functor for inclusions of open sets

$i : \mathcal{U} \to M$ inclusion

$i_* : \Omega^\bullet_c(\mathcal{U}) \longrightarrow \Omega^\bullet_c(M)$

$\omega \longmapsto$ extend ω by zero.

given $U \cup V \leftarrow U \amalg V \underset{i_V}{\overset{i_U}{\leftleftarrows}} U \cap V$

$$0 \leftarrow \Omega^\bullet_c(U\cup V) \xleftarrow{\text{sum}} \Omega^\bullet_c(U)\oplus\Omega^\bullet_c(V) \xleftarrow{(i_{U*},\, i_{V*})} \Omega^\bullet(U\cap V) \leftarrow 0$$

(Mayer-Vietoris) $\qquad (-i_{U*}\omega,\ i_{V*}\omega) \leftarrow\!\shortmid\ \omega$

<u>claim:</u> is exact. $\qquad \rho_U + \rho_V = 1.$

given $\Omega^\bullet_c(U\cap V) \ni \omega = \underbrace{\rho_U\omega}_{\in\Omega_c(U)} + \underbrace{\rho_V\omega}_{\in\Omega_c(V)}$

$\Rightarrow$ long exact seq. on $H^\bullet_c$

$$\cdots \leftarrow H^{q+1}_c(U\cup V) \leftarrow H^{q+1}_c(U)\oplus H^{q+1}_c(V) \leftarrow H^{q+1}_c(U\cap V) \xleftarrow{d_*} H^q_c(U\cup V) \leftarrow \cdots$$

3 Pairings on cohomology

① wedge product $\omega \wedge \eta$

$p + q = p+q$

on M oriented

$$\Lambda^k \times \Omega_c^{n-k} \longrightarrow \mathbb{R}$$

$$(\omega, \eta) \longmapsto \int_M \omega \wedge \eta \in \mathbb{R}$$

claim: is nondegenerate

(ie $\int \omega \wedge \eta = 0 \ \forall \eta \Rightarrow \omega = 0.$)

given $\omega \in \Omega^k$ $\quad \int \omega \wedge \cdot : H_c^{n-k} \longrightarrow \mathbb{R}$

ie an element of $(H_c^{n-k})^*$.

Moreover: well-defined on cohomology.

b/c of Leibniz property of d.

i.e. $H^k(M) \cong \left(H_c^{n-k}(M)\right)^*$.

[arrows are reversed.]

why non-degeneracy:

take $M = U \cup V$.

$$
\begin{tikzcd}
\cdots \arrow[r, "d^*"] & H^q(U\cup V) \arrow[r, "\text{restrict}"] & H^q(U)\oplus H^q(V) \arrow[r, "\text{diff}"] & H^q(U\cap V) \arrow[r, "d^*"] & H^{q+1}(U\cup V) \\
& \otimes & \otimes & \otimes & \otimes \\
& H_c^{n-q}(U\cup V) \arrow[l] \arrow[d, "\int_{U\cup V}"] & H_c^{n-q}(U)\oplus H_c^{n-q}(V) \arrow[l] \arrow[d, "\int_U+\int_V"] & H_c^{n-q}(U\cap V) \arrow[l] \arrow[d, "\int_{U\cap V}"] & H_c^{n-q-1}(U\cup V) \arrow[l, "d_*"'] \quad \cdots \\
& \mathbb{R} & \mathbb{R} & \mathbb{R} &
\end{tikzcd}
$$

(A linear map $A\otimes B \to \mathbb{R}$ is ... $A \to B^*$.)

$$
\begin{tikzcd}
\cdots \arrow[r, "d^*"] & H^q(U\cup V) \arrow[r, "\text{restrict}"] \arrow[d, "\int_{U\cup V}"] & H^q(U)\oplus H^q(V) \arrow[r, "\text{diff}"] \arrow[d] & H^q(U\cap V) \arrow[r, "d^*"] \arrow[d] & H^{q+1}(U\cup V) \arrow[d] \\
\arrow[r] & \left(H_c^{n-q}(U\cup V)\right)^* \arrow[r] & \left(H_c^{n-q}(U)\right)^* \oplus \left(H_c^{n-q}(V)\right)^* \arrow[r] & \left(H_c^{n-q}(U\cap V)\right)^* \arrow[r, "(d_*)^*"] & \left(H_c^{n-q-1}(U\cup V)\right)^*
\end{tikzcd}
$$

claim: ① commute. ② Poincaré lemmas: $H^*(U) \cong H^*(\mathbb{R}^n)$

5 lemma $\Longrightarrow$ $H^q(M) \cong (H_c^{n-q}(M))^*$. (PD)

<u>induction idea</u>: suppose true $\forall\, M = \mathcal{U}_0 \cup \dots \cup \mathcal{U}_{p-1}$

a <u>good</u> cover by <u>p open sets</u>

$\uparrow$ all $\mathcal{U}_\alpha$, $\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta$, $\dots \mathcal{U}_{\alpha\beta\gamma} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$. ... are $\cong$ <u>ball</u>

Consider $M = \mathcal{U}_0 \cup \dots \cup \mathcal{U}_p$

$V \equiv (\mathcal{U}_0 \cup \dots \cup \mathcal{U}_{p-1}) \cap \mathcal{U}_p$ has a good cover by p open sets.

PD holds for $\left\{\begin{array}{l} \mathcal{U}_0 \cup \dots \mathcal{U}_{p-1} \\ \mathcal{U} \equiv \mathcal{U}_p \\ V \\ \mathcal{U} \cap V \end{array}\right.$ $\xrightarrow{\text{5-lemma}}$ PD holds for M.

(If M is compact can drop the c.)

$\blacksquare$

② closed p-forms and p-cycles

$\omega \in \Omega^p(M)$ $\qquad$ $\mu \in \Omega_p(M)$ $\qquad$ $\partial\mu = 0$

$d\omega = 0$.

$$H^p(X) \times H_p(X) \longrightarrow \mathbb{R}$$

$$([\omega], [\mu]) \longmapsto \int_\mu \omega$$

$$\int_{\mu+\partial\nu} (\omega + d\eta) = \underline{\int_\mu \omega} + \underbrace{\int_{\partial\nu} \omega}_{= \int_\nu d\omega = 0} + \underbrace{\int_\mu d\eta}_{= \int_{\partial\mu} \eta = 0} + \underbrace{\int_{\partial\nu} d\eta}_{= \int d^2\eta = 0}$$

is well-def'd on $H^\bullet \times H_\bullet$.

③ on an oriented manifold M

closed k-dim'l submanifold $S \subset M$

$\partial S = 0$.

$$[S] \in H_k(M)$$

$i : S \to M$ inclusion.

$i^* : \Omega^\bullet(M) \to \Omega^\bullet(S)$ pullback = restriction

$\forall\, \omega \in H^k(M)$

$$\int_S i^*\omega \equiv \underline{\int_M \omega \wedge \eta_S}$$

For each $[S] \in H_k(M)$

$\rightsquigarrow [\eta_S] \in H^{n-k}(M)$

(nondenerate by P.D.) $\Rightarrow$ $\boxed{H_k(M) \cong H^{n-k}(M).}$

$$b_k(M) \overset{P.D.}{\cong} b_{n-k}(M) \overset{P.D.}{\cong} b^k(M) \cong b^{n-k}(M).$$

(for oriented M)

$\uparrow$ M compact

(Note: no torsion)

between A & B

A pairing (ω, η) is nondegenerate $\Leftrightarrow$ $\cdot\ (\omega,\eta)=0\ \forall \eta \Rightarrow \omega=0$ and $\cdot\ (\omega,\eta)=0\ \forall \omega \Rightarrow \eta=0$

$\Leftrightarrow A \cong B^*$

A, B are v.s. over $\mathbb{R}$

$$(a, b) = a_\alpha M_{\alpha i} b_i$$

a linear pairing

nondegenerate $\Longleftrightarrow$ $\det M \neq 0$

$$H^q_c(\mathbb{R}^n) = \delta^{q,n}\, \mathbb{R} = \langle dx^1 \wedge \dots \wedge dx^n \rangle$$

$$H^q(\mathbb{R}^n) = \delta^{q,0}\, \mathbb{R}$$

$$= \langle \text{constant f'n} \rangle$$

f w/ $df = 0$

Warning [B&Tu]

$H_k \qquad H^{n-k}$

$$S \rightarrow \eta_S$$

$$S \rightarrow [\eta_S'] \in H_c^{n-k} \qquad \eta_S' \neq \eta_S$$

S gives an element

$$\text{of } (H^k)^* \cong H_c^{n-k}$$

Wen 2018

NLSM & topological order.

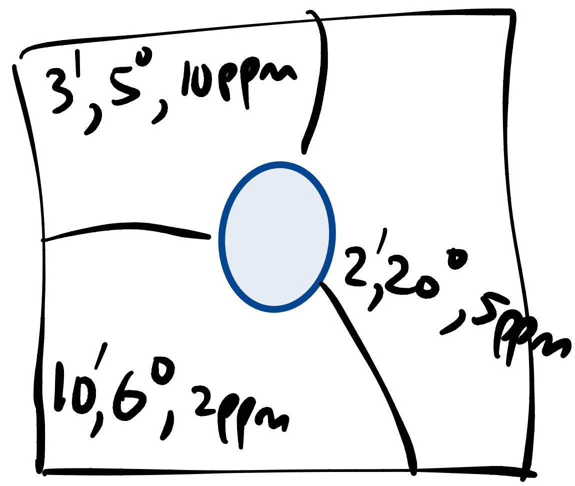
π_k(target space) $\to$ gauge group.

2.6 Čech cohomology

logical extreme of M-V idea.

Parable #1

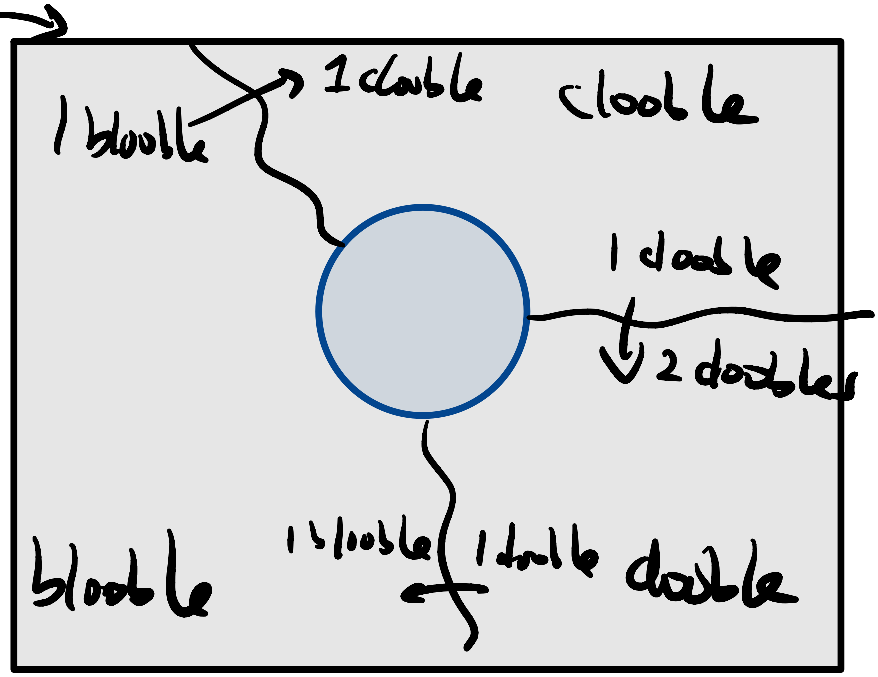
preferences: hot > cold
dry > wet
high > low



Q: is there a fn on this space whose max is where you should live?

(No.)

transition f'ns



flux = Π (exchanges)

= holonomy

= factor by which your investment changes.

currency exchange $\longleftrightarrow$ gauge transformation.

<u>Replace</u>:

clooble $\longrightarrow$ cookie

doobles $\longrightarrow$ donut

bloobles $\longrightarrow$ bagel.

Intrinsic value of baked goods $\longleftrightarrow$ higgs field.

obviousness of stupidity of exchange rates $\leftrightarrow$ mass of vector boson

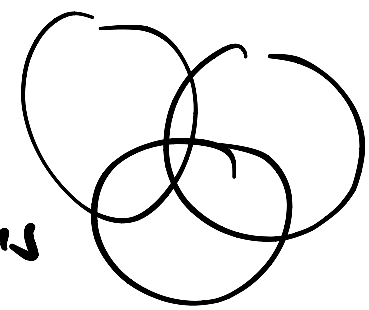
Čech cohomology (simplest version)

Cover M w/ open sets

$$M = U_0 \cup \dots \cup U_p$$

$$\mathcal{U}_{\alpha\beta} \equiv U_\alpha \cap U_\beta \quad , \quad \mathcal{U}_{\alpha\beta\gamma} \equiv \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$$

$\dots$

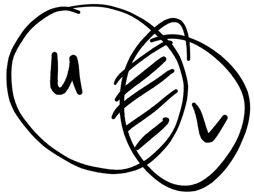


let

$$C^k \equiv \left\{ \underline{\text{lo}}\text{cally } \underline{\text{con}}\text{stant functions } \coprod_{\alpha_0 \dots \alpha_k} \mathcal{U}_{\alpha_0 \dots \alpha_k} \longrightarrow A \right\}$$

disj. union

abelian group.

$\underline{\text{eg}}$: $M =$ 

$f \in C^0(\{U,V\})$

is $\begin{cases} f(U) \in A \\ f(V) \in A. \end{cases}$

$\omega \in C^1(\{U,V\})$ is $\omega(U \cap V) \in A$.

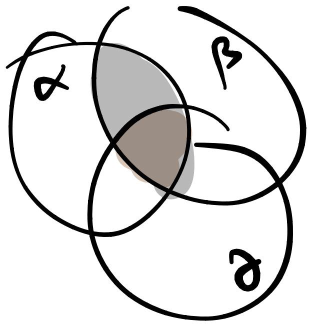
Coboundary map : $\delta : C^k \to C^{k+1}$

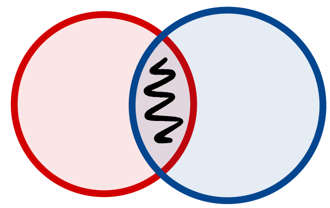
difference of restriction.

given $f : \mathcal{U}_{\alpha\beta} \to A$

we can define

$f\big|_{\alpha\beta\gamma} : \mathcal{U}_{\alpha\beta\gamma} \to A.$





$\delta : C^0 \to C^1$

$f \mapsto (\delta f)_{\alpha\beta} = f_\alpha - f_\beta$

checks agreement

$\delta : C^1 \to C^2$

$f \mapsto (\delta f)_{\alpha\beta\gamma} = f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha}$

Assume : $f_{\alpha\beta} \equiv -f_{\beta\alpha}$.

and $f_{\alpha_1 \cdots \alpha_n} = (-1)^\sigma f_{\alpha_{\sigma_1} \cdots \alpha_{\sigma_n}}$ $\sigma \in S_n$.

$$(\delta f)_{\alpha_0 \ldots \alpha_k} = \sum_i (-1)^i f_{\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_k}$$

missing

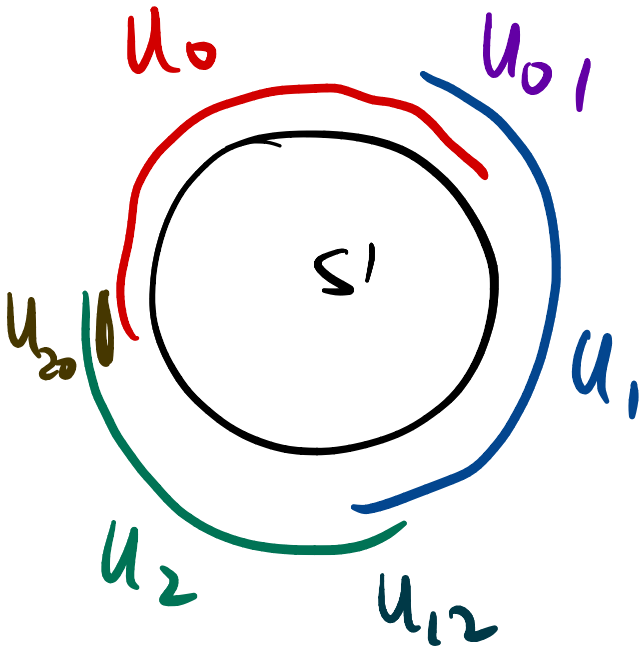
choose an order of $\mathcal{U}_\alpha$.

$\boxed{\text{claim: } \delta^2 = 0.}$

$\longrightarrow \check{H}^\bullet(\mathcal{U})$

open cover

eg:



$C^0 = \{ \omega_\alpha, \alpha = 0,1,2.$ ω_α is constant on $\mathcal{U}_\alpha \}$

$\cong A^3$

$C^1 = \{ \eta_{\alpha\beta} \mid \eta_{\alpha\beta} \text{ is const on } \mathcal{U}_{\alpha\beta} \} \cong A^3$

$$0 \longrightarrow A^3 \xrightarrow{\delta} A^3 \longrightarrow 0$$

$$\delta = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}. \quad \text{rank } 2.$$

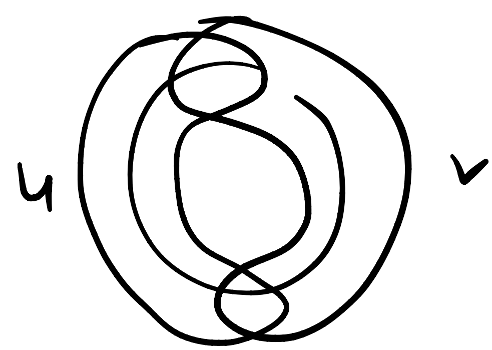
$$H^0(S^1) = \ker \delta = \{\omega_0 = \omega_1 = \omega_2\} \cong A$$

$$H^1(S^1) = A^3/\operatorname{im}\delta = \langle (g,0,0) \rangle \cong A$$

$$\operatorname{im}\delta = \{\eta \mid \eta_{01} + \eta_{12} + \eta_{20} = 0\}$$

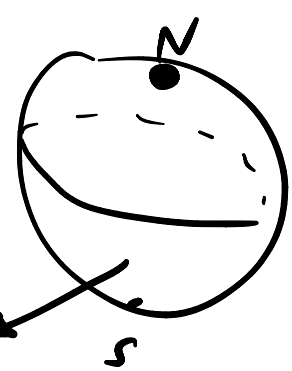
This was a good cover

(bad cover

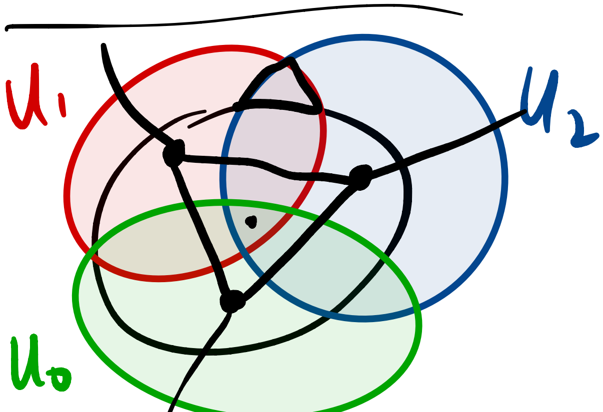


$U \cap V = 2$ balls)

A good cover of S^2:



U_3 = northern hemisphere



$$\longrightarrow C^\bullet \longrightarrow \check{H}^\bullet$$

open sets $U_\alpha \longleftrightarrow$ 0-cells α

$U_{\alpha\beta} \longleftrightarrow$ 1-cell w/ $\partial e_{\alpha\beta} = \alpha - \beta$

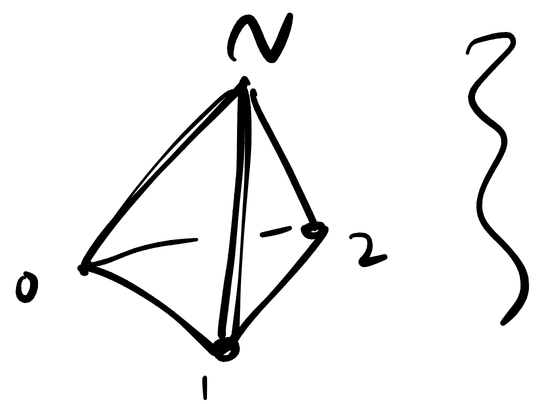
$U_{\alpha\beta\gamma} \longleftrightarrow$ 2-cell w/ $\partial \omega_{\alpha\beta\gamma} = e_{\alpha\beta} + e_{\beta\gamma} + e_{\gamma\alpha}$.

$\vdots$

Claim:

In n dims no $n+2$-overlaps of a good cover. $\Rightarrow C^{n+1} = \emptyset$.

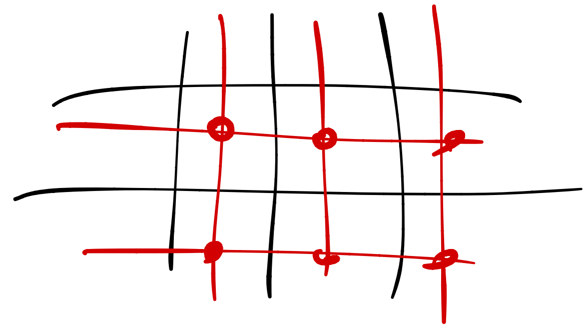
eg



Hodge: $* dA = d\tilde{A}$

$\uparrow$ on spacetime.

— Involves metric



$$\underset{p}{\int \omega} \wedge \underset{n-p}{\eta}$$ does not.

4d Abelian gauge theory

U(1)

$$S[A] = -\frac{1}{4g^2}\int_{X_4} F\wedge *F + \frac{\theta}{16\pi^2}\int_{X_4} F\wedge F$$

A well-defined

$\Rightarrow \quad dF = 0$

$[F] \in H^2(X)$

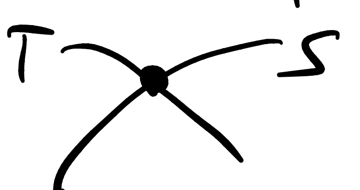
$$H^2(X_4) \simeq \left(H^2(X_4)\right)^*$$

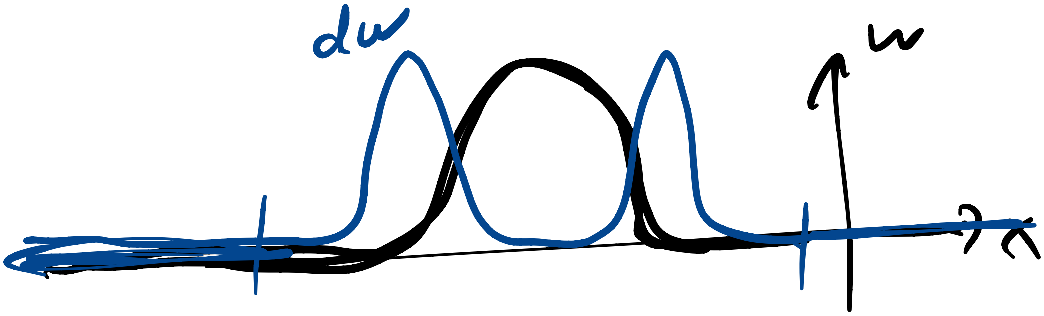
intersection pairing

$$\int [\eta_S]\wedge[\eta_T] = \#(S\cap T)$$

$\uparrow$ k-cycle $\quad \uparrow$ (n-k)cycle

intersection #





$\{x \mid \omega(x) \neq 0\}$ is compact

$$d : \underline{\Omega_c^p} \to \Omega_c^{p+1}$$

$$\underline{\Omega_c^p(M)}$$

$$A \otimes B \to \mathbb{R}$$
$$a, b \mapsto (a,b) = a_\alpha M_{\alpha i} b_i$$

If $\det M \neq 0$

then $A \cong B^*$

If $Mv = 0$

v is not detected

by $\underline{(a,v) = 0}$

$M : A \to B^*$

has a kernel is $\det M = 0$.

$$Q = Q_1 + Q_2 \quad \text{on} \quad \Omega^\bullet$$

if $\{Q_1, Q_2\} = 0$ ...

$$H^\bullet_Q(\Omega) = H^\bullet_{Q_1}\Big(\underbrace{H^\bullet_{Q_2}}\Big)$$

tic tac toe lemma

[Bott & Tu.]

or [Schoutens, Huijse]