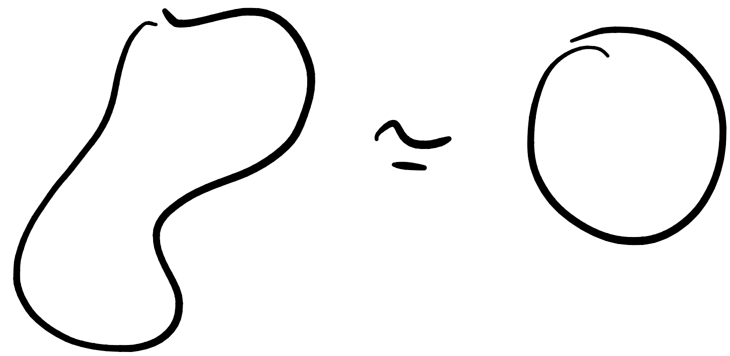
Homotopy equivalence & (co)homology

$X \simeq Y$.



Recall: $X \simeq Y \iff \exists\, f : X \to Y$ and $g : Y \to X$

s.t. $f \circ g \simeq \mathbb{1} \simeq g \circ f$

"is homotopic to"

Two maps $f_{0,1} : M \to N$ are homotopic if $\exists$ continuous $F : M \times I \to N$

$F(x,0) = f_0(x)$

$F(x,1) = f_1(x)$

Fact: If $X \simeq Y$ then $H^\bullet_{dR}(X) \cong H^\bullet_{dR}(Y)$.

Lemma: Homotopic maps induce the same map on cohomology.

Pf: $f_{0,1} : M \to N$

$F : M \times \mathbb{R} \to N$ w/ $F(x,t) = f_t(x)$
$t = 0, 1$.

Let: $s_{0,1} : M \to M \times \mathbb{R}$ 0-section
1-section

$$f_t = F \circ s_t \quad (t = 0, 1)$$

$$\begin{tikzcd}
M \times \mathbb{R} \arrow[d, shift left, "\pi"] \arrow[drr, "F"] & & \\
M \arrow[u, shift left, "s_{0,1}"] \arrow[rr, "f_{0,1}"'] & & N
\end{tikzcd}$$

$$f_t^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$$

$$f_t^* = (F \circ s_t)^*$$
$$= s_t^* \circ F^*$$

$$\left(\begin{tikzcd}
\Omega^\bullet(M \times \mathbb{R}) \arrow[d, "s_{0,1}^*"'] & & \\
\Omega^\bullet(M) & & \Omega^\bullet(N) \arrow[ull, "F^*"'] \arrow[ll, "f_{0,1}^*"]
\end{tikzcd}\right)$$

Recall: $s_{0,1}^* : H^\bullet(M\times\mathbb{R}) \to H^\bullet(M)$

is an isomorphism.

$$s_0^* = (\pi^*)^{-1} = s_1^* .$$

$$\Rightarrow f_0^* = F\circ s_0^* = F\circ s_1^* = f_1^* \quad \blacksquare$$

$\Rightarrow$ If $X \simeq Y \Rightarrow H^\bullet_{dR}(X) \cong H^\bullet_{dR}(Y)$.

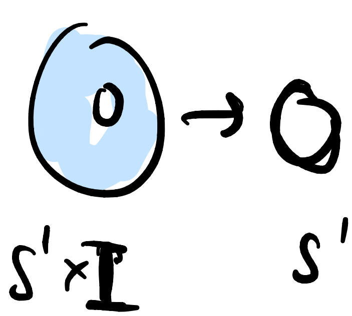
$\exists\ f : X \to Y$ $\quad g\circ f = f_0 \simeq \mathbb{1} = f_1$

$g : Y \to X$

$$\Rightarrow f_0^* \circ g_0^* = \mathbb{1} \text{ on } H^\bullet(X), \quad g_0^*\circ f_0^* = \mathbb{1} \text{ on } H^\bullet(Y)$$

$$f_0^* : H^\bullet(X) \xrightarrow{\cong} H^\bullet(Y).$$

If A is a def. retract of X

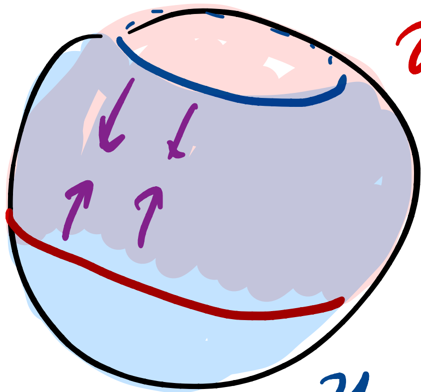


then $H^\bullet_{dR}(X) \cong H^\bullet_{dR}(A)$.

e.g. $H^\bullet(n\text{-ball}) \cong H^\bullet(\mathbb{R}^n) \cong H^\bullet(pt)$.

eg:

S^n:



$\mathcal{U}_N \cap \mathcal{U}_S = S^{n-1} \times I \simeq S^{n-1}$

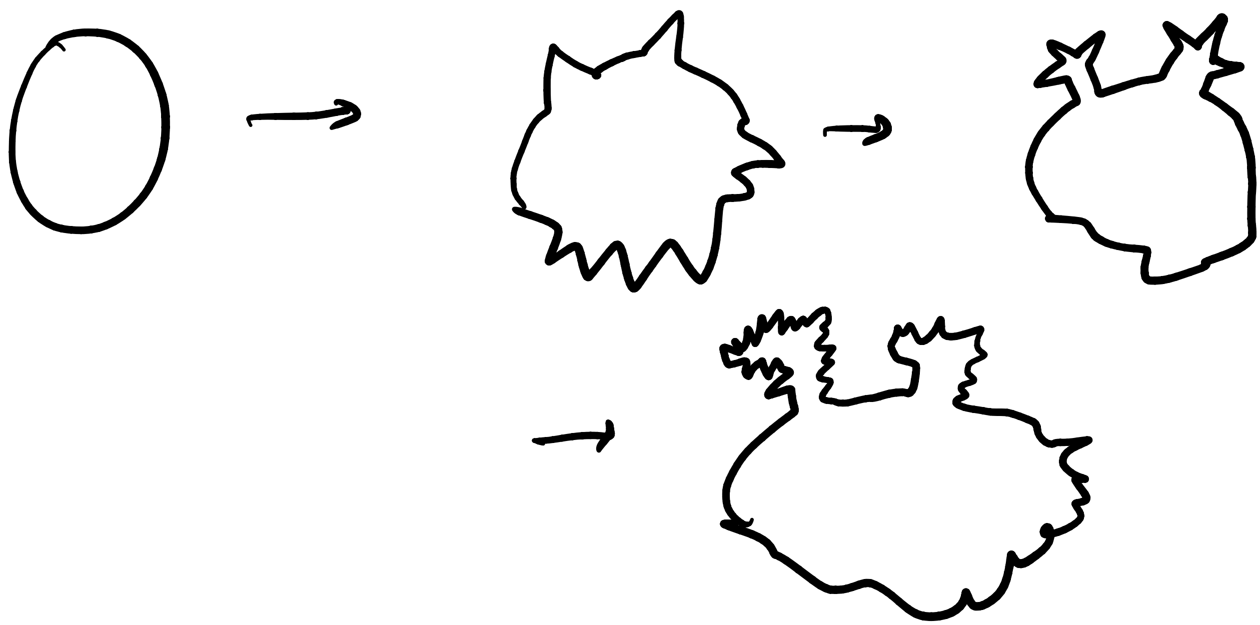
$$\begin{tikzcd} \cdots \arrow[r] & H^q(S^n) \arrow[r, "\mathrm{res}"] & H^q(\mathcal{U}_N) \oplus H(\mathcal{U}_S) \arrow[r] & H^q(S^{n-1}) \arrow[r] & H^{q+1}(S^n) \arrow[r] & \cdots \end{tikzcd}$$

determines $H^q(S^n)$ from $H^q(S^{n-1})$.

WORRY: f^* assumes $f \in C^\infty$.

what if $X \simeq Y$ by a map f continuous but not C^∞?

Thm: any continuous map is homotopic to a smooth map.



Homotopy eq & homology.

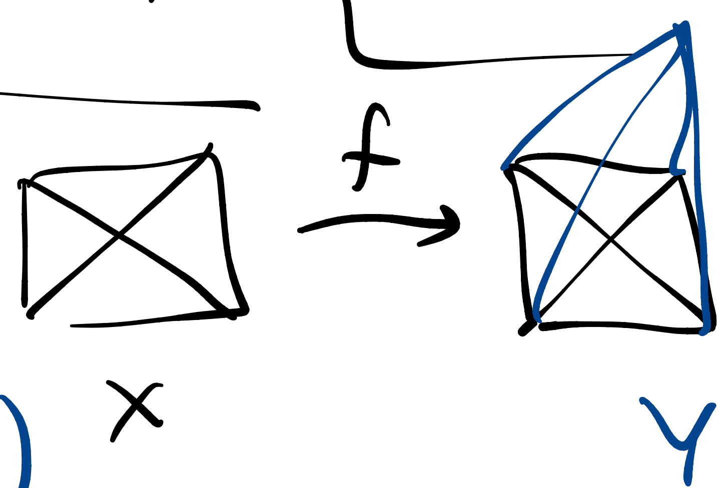
$$H^q_{dR}(X) \cong H_{n-q}(X, \mathbb{R})$$ has less info

then $H_{n-q}(X, \mathbb{Z})$

to show: $X \simeq Y$

then $H_\bullet(X, \mathbb{Z}) \cong H_\bullet(Y, \mathbb{Z})$.

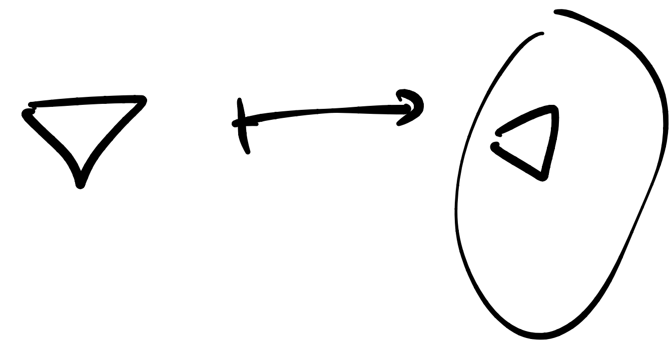
$f: X \to Y$.



choose a cellulation of Y
compatible w/ f(cellulation of X)

remark: singular homology avoid this.

chains $= \left\{ \text{continuous maps } \Delta_q \to X \right\}$

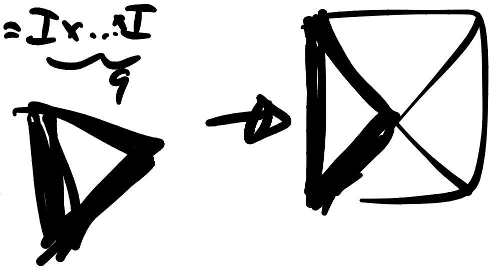


think of a cell of X as a continuous map

$$\sigma_q : \Delta_q \longrightarrow X .$$

$\uparrow$ q-ball, $I^q = \underbrace{I \times \cdots \times I}_{q}$

"characteristic map"

σ_q : 

$$f_\# : \Omega_\bullet(X) \to \Omega_\bullet(Y)$$

$$\sigma \longmapsto f_\#(\sigma) = f \circ \sigma : \Delta_q \to Y$$

is a chain map: $f_\# \partial = \partial f_\#$.

$$\Rightarrow \quad f_* : H_\bullet(X) \to H_\bullet(Y)$$
$$[\sigma] \mapsto [f_\#(\sigma)]$$
$$= [f \circ \sigma].$$

$H_\bullet$: manifolds $\longrightarrow$ abelian groups

covariant functor.

Lemma: If $f_{0,1} : X \to Y$ are homotopic then $(f_0)_* = (f_1)_* : H_\bullet(X) \to H_\bullet(Y)$.

$\Rightarrow$

Main Result: $f : X \to Y$ $\quad f \circ g \simeq \mathbb{1}$
$g : Y \to X$ $\quad g \circ f \simeq \mathbb{1}$.

$$(fg)_* = f_* g_* \text{ and } \mathbb{1}_* = \mathbb{1}$$

$$f_* g_* = \mathbb{1} = g_* f_*$$

$f_* : H_\bullet(X) \to H_\bullet(Y)$ is an isomorphism.

Pf of lemma: Given $F: X \times I \to Y$

$$F(x,0) = f_0(x), \quad F(x,1) = f_1(x).$$

construct a homotopy operator

$$K : \Omega_q(X) \to \Omega_{q+1}(Y)$$

s.t. $$K\partial + \partial K = f_{0\#} - f_{1\#} \quad \text{on } \Omega_q(X).$$

act on a q-cycle $\alpha \in \Omega_q(X)$ s.t. $\partial \alpha = 0$

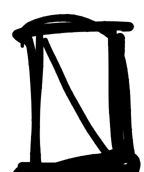
$$\Rightarrow \partial(K\alpha) = (f_{0\#} - f_{1\#})(\alpha)$$

$$[\text{BHS}] \Rightarrow 0 = [f_{0\#}(\alpha)] - [f_{1\#}(\alpha)]$$

$$= f_{0*}[\alpha] - f_{1*}[\alpha].$$

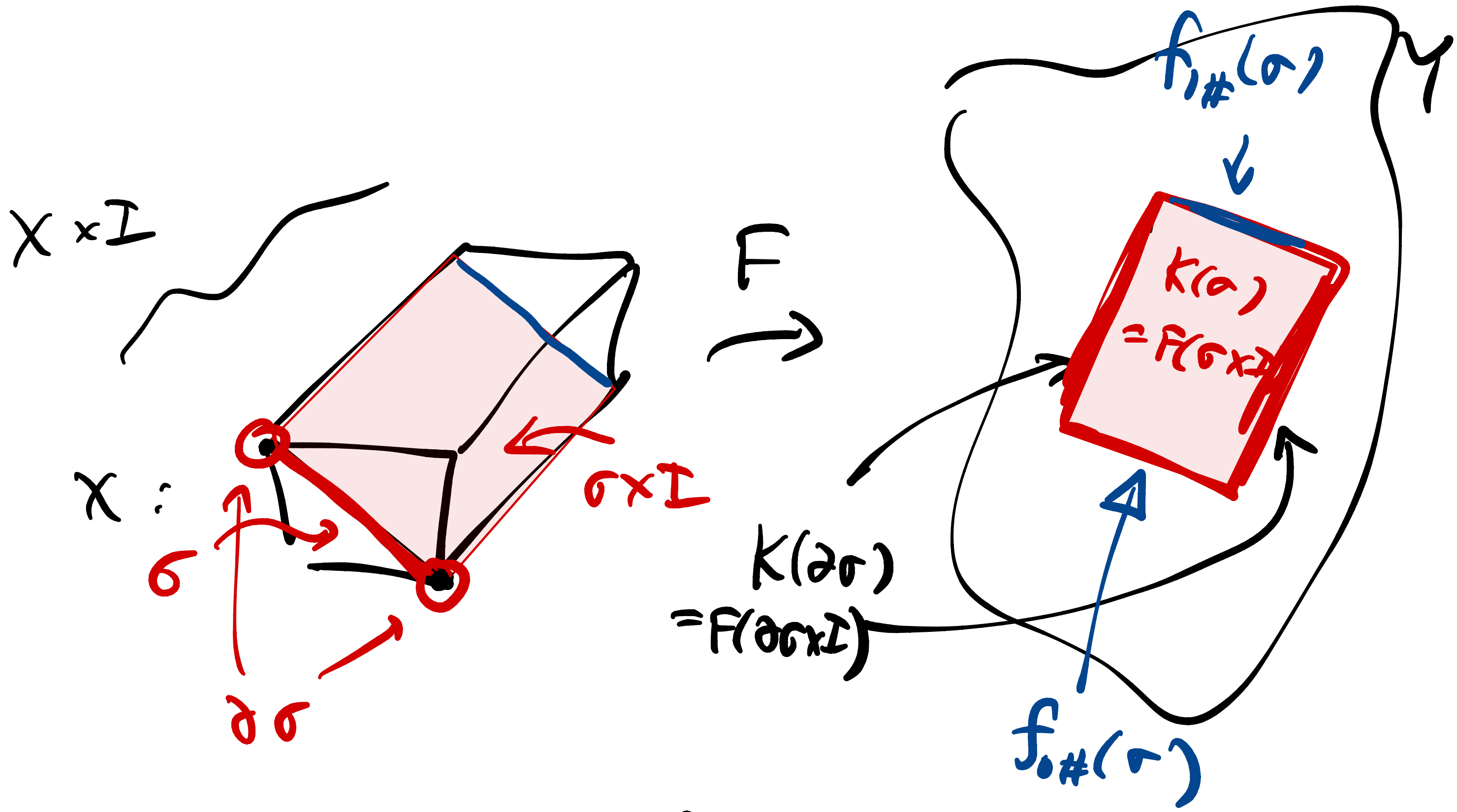
who is K?

idea: cellulation of X $\Rightarrow$ cellulation of $X \times I$.

I 

X

Def: $\boxed{K(\sigma) = F(\sigma \times I)}$



$$\partial K(\sigma) = K(\partial\sigma) + f_{1\#}(\sigma) - f_{0\#}(\sigma)$$

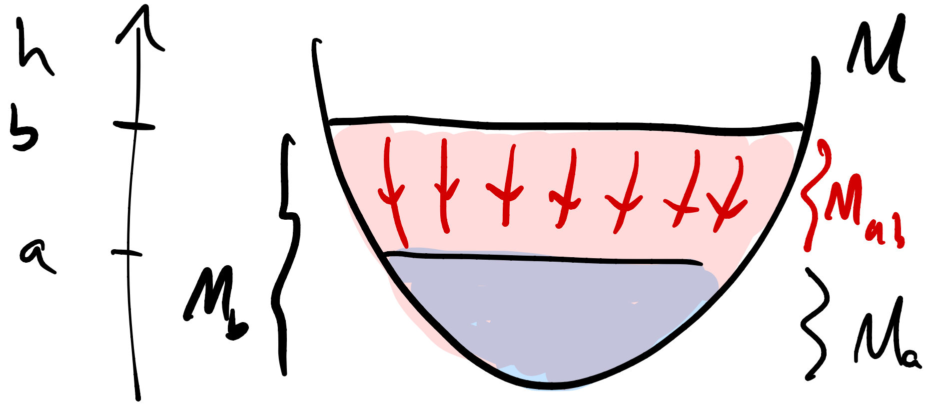
■

3.4 Morse theory & homotopy equivalence.

Suppose h is a Morse f'n on $_{\wedge}^{\text{smooth}}$ M.

Let $M_a \equiv h^{-1}([-\infty, a])$

If $h^{-1}([a,b]) = M_{ab}$ is compact and contains no critical pts then $M_a \simeq M_b$



<u>Pf</u>: put a metric γ_{ij} on M.

<u>gradient</u>: $\gamma_{ij} \nabla^i f \, Y^j \equiv \langle \vec{\nabla} f, \vec{Y} \rangle$

$$\equiv df(\vec{Y}).$$

$\forall Y$ vector field on X

let $\vec{X} = \dfrac{-\vec{\nabla} h}{\|\vec{\nabla} h\|}$ $\quad \left(\|\vec{Y}\| = \sqrt{\langle Y, Y \rangle} \right)$

is a unit vec. field, well-def'd away from critical pts (where $\|\vec{\nabla} h\| = 0$)

Flow lines of $\vec{X}$ in M_{ab} specify a deformation retraction $M_b \to M_a$.

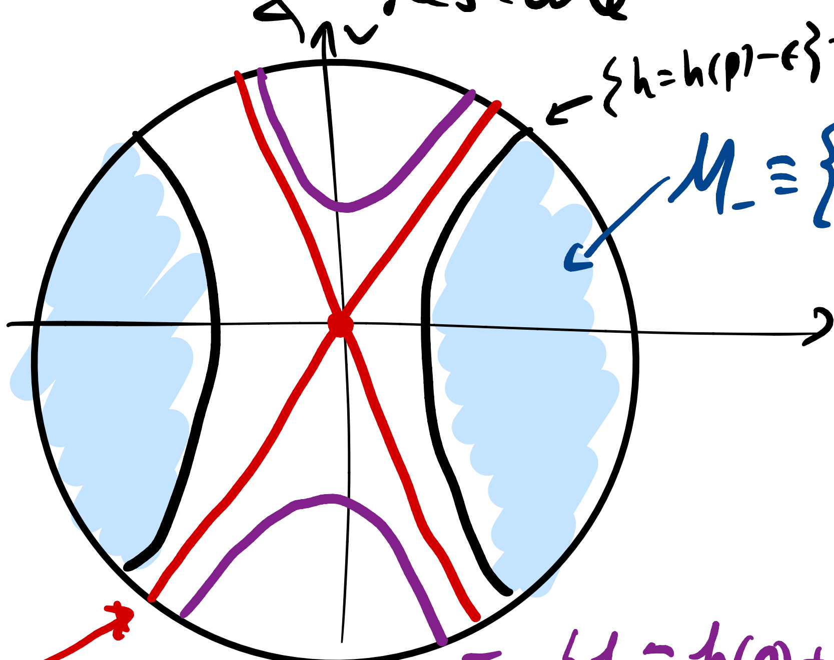
Suppose $h^{-1}([a,b])$ is compact with <u>one</u> critical pt of Morse index k.

Then $\underline{M_b \simeq M_a \cup e^k}$ ← a <u>k-cell</u>.

<u>idea:</u> $\exists$ coords on M near the critical pt p s.t.

$$h = h(p) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2 \quad \cancel{+O(x^3)}$$

(<u>pf</u>: diagonalize $\partial_i \partial_j h|_p$ & rescale)

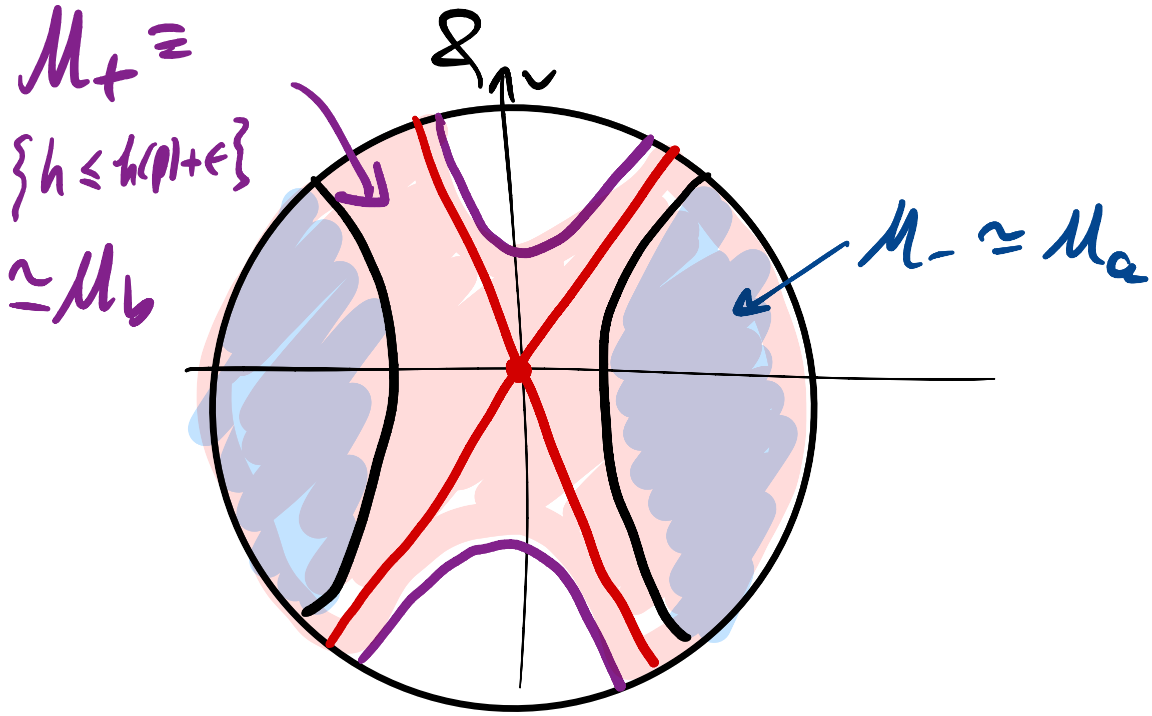


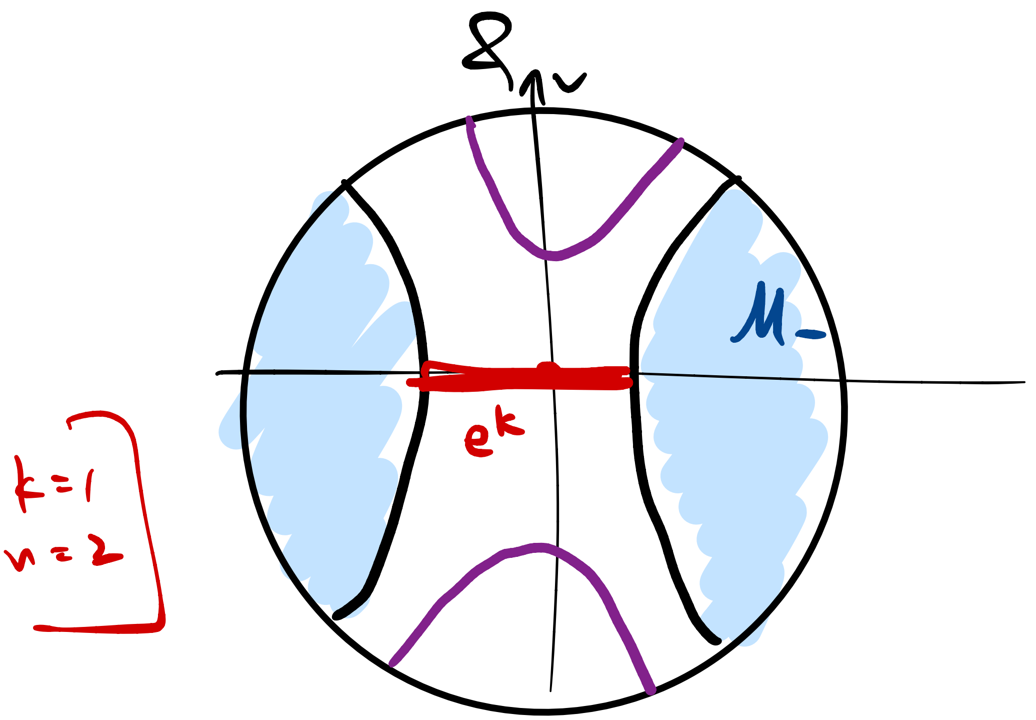
$n = 2$
$\underline{k = 1}$.

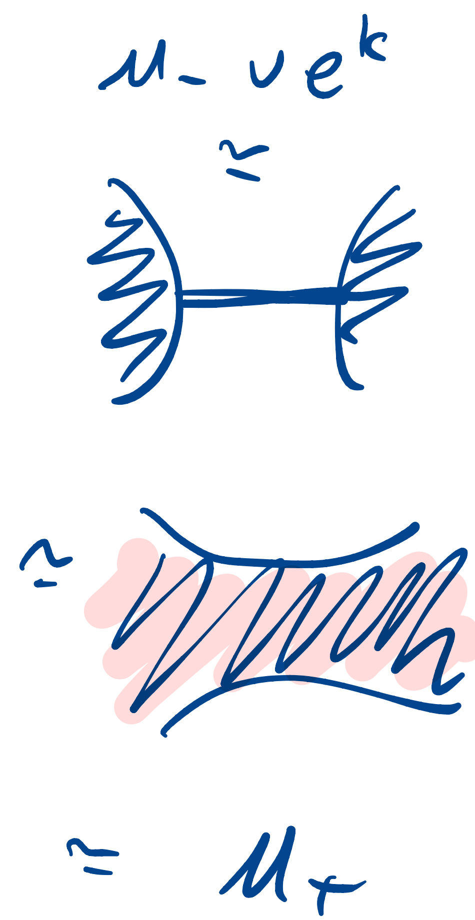
$h = h(p) - u^2 + v^2$

$\underline{\{u^2 = v^2\}} =$
$\{h = h(p)\}$

$\{h = h(p) + \epsilon\} = \{u^2 = v^2 - \epsilon\}$.

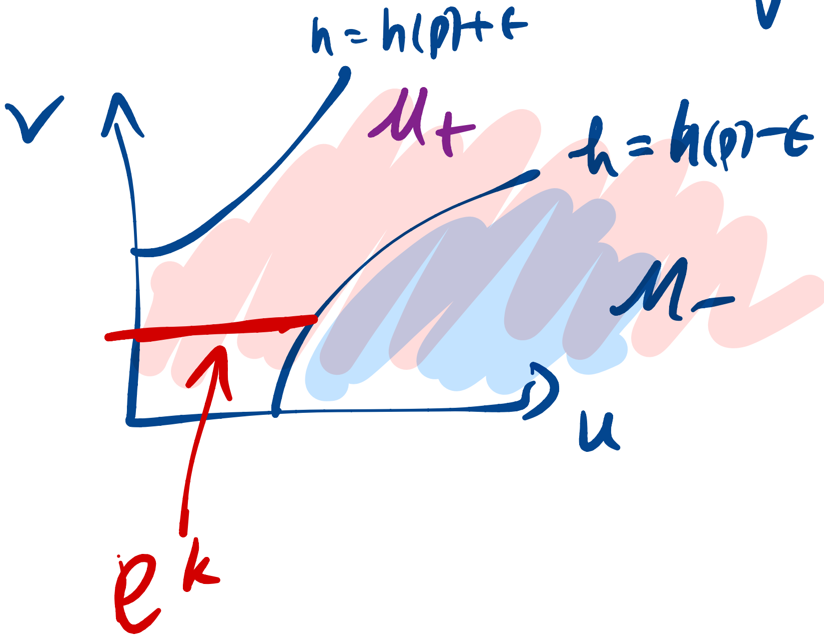

$\mathcal{M}_+ \simeq$
$\{h \leq h(p)+\epsilon\}$
$\simeq \mathcal{M}_b$
$\mathcal{E}_v$
$\mathcal{M}_- \simeq \mathcal{M}_a$


$\mathcal{E}_v$
$\mathcal{M}_-$
e^k
$k=1$
$n=2$


$\mathcal{M}_- \cup e^k$
$\simeq$
$\simeq$
$\simeq \mathcal{M}_+$

general n,k: let $u^2 \equiv x_1^2 + \dots x_k^2$

$v^2 \equiv x_{k+1}^2 + \dots x_n^2$



$\exists$ Morse f'ns $\Rightarrow$ any compact mfld has a finite cell decomposition.

Milnor, Morse Theory for More.

$$\left[\frac{\mathrm{tr} F \wedge F}{16\pi^2} = c_2\right] \in H^4(X, \mathbb{Z})$$

$\pi_3(G)$

3.5 Homotopy Groups

Let X be a topological space

w/ a base point, $p \in X$

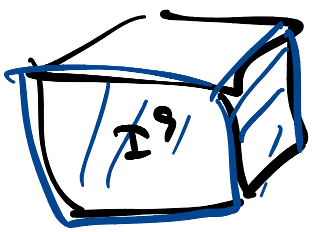
Homotopy groups of X are

$$\boxed{\pi_q(X) \equiv \pi_q(X, p) \equiv \text{homotopy classes of maps: } (I^q, \partial I^q) \to (X, p)}$$

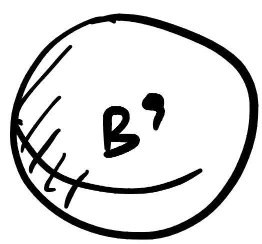
$$I^q \equiv \underbrace{I \times I \times I \cdots I}_{q \text{ times}}$$

$\simeq$ a map:

$$(B^q, \partial B^q = S^{q-1}) \to (X, p)$$

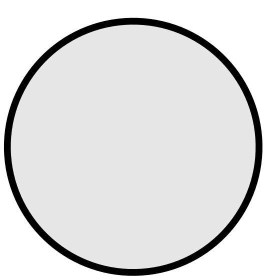


$\simeq$



$\partial B^q = S^{q-1}$

$$I^q / \partial I^q \simeq S^q$$



$$B^2 / \partial B^2 = S^2$$

alternatively:

$$\pi_q(X,p) = \text{maps} : (S^q, N) \to (X,p) \Big/ \text{homotopy.}$$

For $q>0$ $\pi_q(X)$ is a group under:

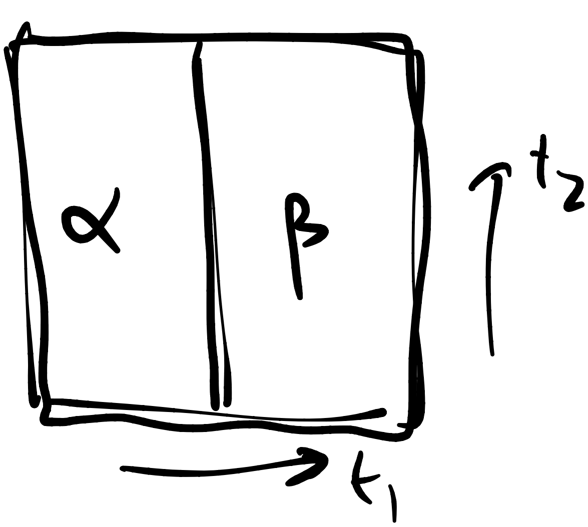
Given $\alpha, \beta : (I^q, \partial I^q) \to (X, p)$

$$\left([\alpha], [\beta] \in \pi_q(X,p) \right)$$

$$[\alpha][\beta] \equiv [\alpha * \beta]$$

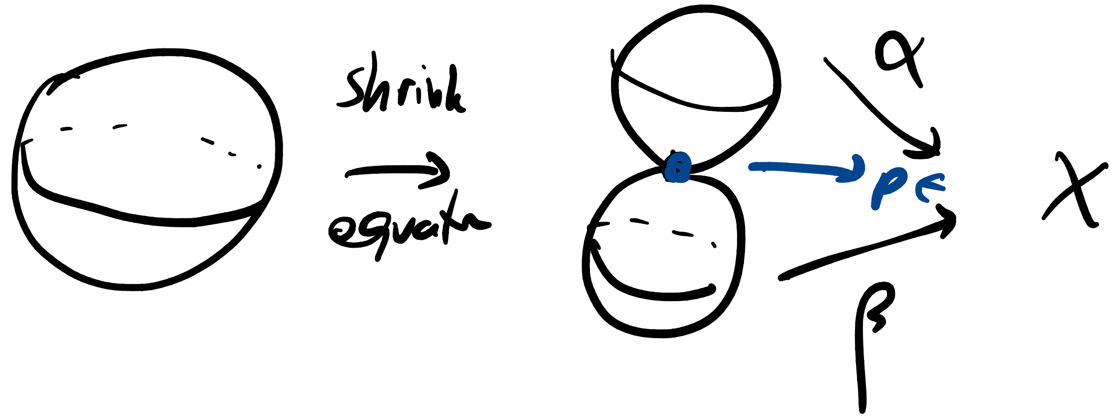
$$(\alpha * \beta)(t_1 \ldots t_q) = \begin{cases} \alpha(2t_1, t_2 \ldots t_q) & 0 \le t_1 \le \frac{1}{2} \\ \beta(2t_1 - 1, t_2 \ldots t_q) & \frac{1}{2} \le t_1 \le 1. \end{cases}$$

$q=2$:



black $\to p$.

OR:

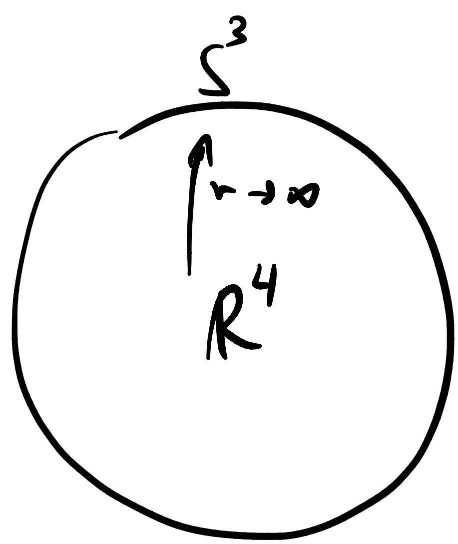


Next: basic properties of $\pi_q(X)$.

$$H_1(X) = \pi_1(X) \Big/ \underbrace{[\pi_1(X), \pi_1(X)]}_{aba^{-1}b^{-1} \sim 1}.$$

$H_q(X)$?? $\pi_q(X)$

$\pi_3(G)$: YM in $\mathbb{R}^4$.



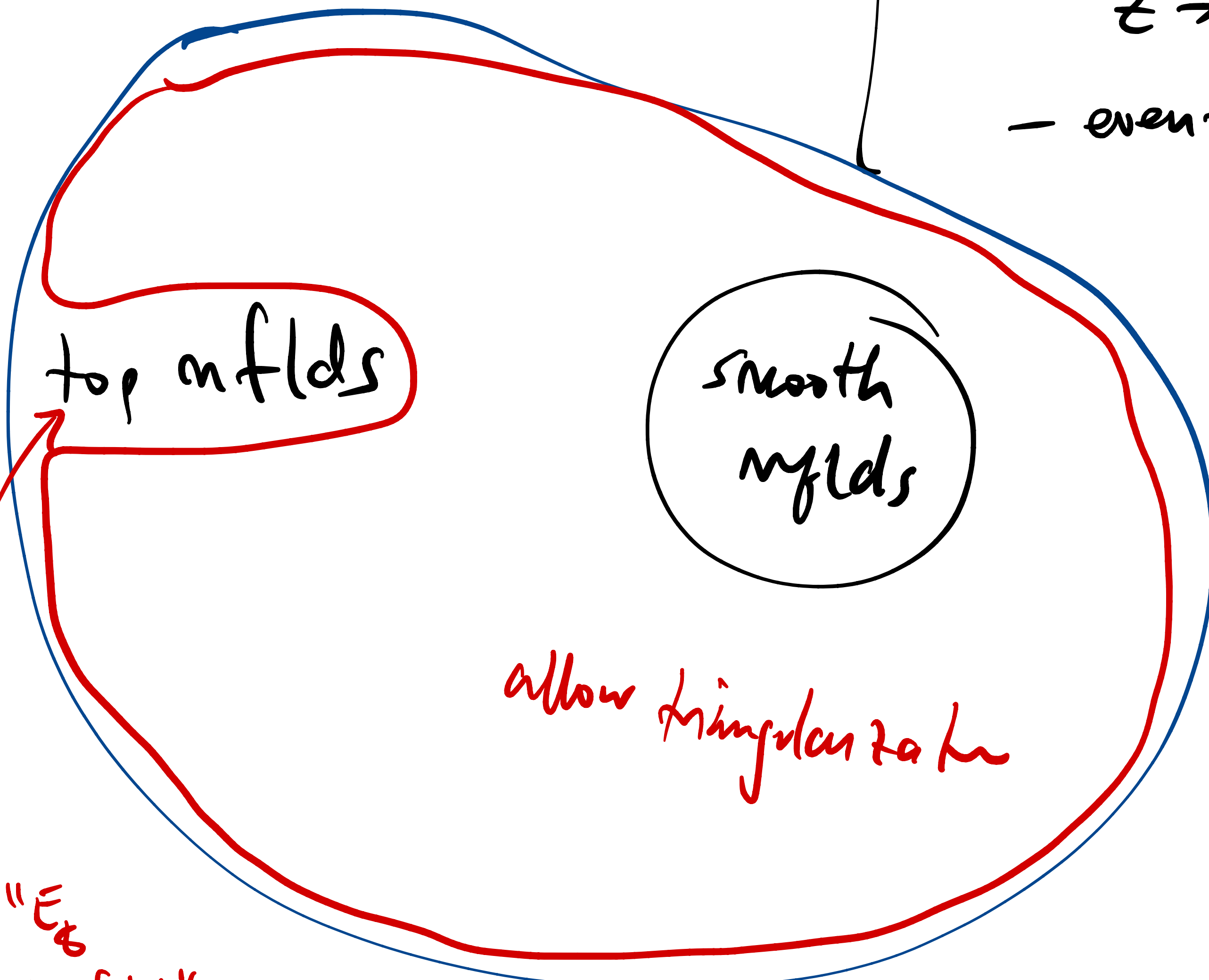
finite action $\Rightarrow$ $A \overset{r\to\infty}{\simeq} g^{-1} dg$

$F \xrightarrow{r \to \infty} 0$

$g : S^3 \to G$ $[g] \in \pi_3(G)$.

$$\frac{\sum_i dz_i \wedge d\bar{z}_i}{\sum |z_i|^2} = \omega$$

- gauge inv of $z \to \lambda z,\ \lambda \in \mathbb{C}^*$
- $U(N+1)$ symmetric $z \to Mz$.
- even-rank.



"E_8 4-Manifold".

$\alpha \in H_2(X,\mathbb{Z})$

$(\alpha,\beta) \in \mathbb{Z}$

$= \#(\alpha \cap \beta)$

$(\alpha,\beta) = (\beta,\alpha)$

$G = ADE$

6d $(2,0)_\wedge$ theory on $M_4 \times \mathbb{R}^{1,1}$

$\boxed{G = U(1)}$

each $\begin{cases} \cdot \text{ SD}^{\wedge} \text{ harmonic 2-form on } M_4 \to \text{left mover} \\ \text{ASD} \quad \text{"} \quad \to \text{right mover} \end{cases}$

$b_2^+ - b_2^- \propto c_-$

intersection form $\to$ K-matrix

K $\qquad \mathcal{L} \ni K_{IJ}\, \partial\phi^I \partial\phi^J$

if $K = K_{E_8}$: edge theory of E_8 2+1 d invertible state.

eg: Hellerman, 2021

a "Information theory of grav. anomalies"

$$S[C] = \frac{1}{4\pi} \int_{M_7 = M_4 \times \mathbb{R}^{2,1}} C \wedge dC$$

$$= \frac{K_{IJ}}{4\pi} \int_{\mathbb{R}^{2,1}} A_I \wedge dA_J$$

if $H_1(M) = \phi$

$$C = \sum_{\alpha_I \in H^2(M)} \alpha_I \wedge A_I$$

$$K_{IJ} = \int_{M_4} \alpha_I \wedge \alpha_J$$

α_I = Poincaré dual of 2-cycle I