Announcement: lectures 19 & 20 next week, usual time & zoom location.

Higher homotopy groups & homology:

For general q $\exists$ a natural homomorphism

$$i : \pi_q(X) \longrightarrow H_q(X)$$
$$[f] \longmapsto \underline{\underline{f_*(u)}}$$

$$f : S^q \to X$$
$$f_* : H_q(S^q) \to H_q(X)$$
$$\simeq \mathbb{Z} = \langle u \rangle$$

neither 1-1 nor onto.

Hurewicz Isomorphism Thm:

If $q > 1$ and $\pi_k(X) = 0$ for $1 \leq k < q$

then $H_k(X) = 0$ for $1 \leq k < q$

and $H_q(X) \cong \pi_q(X)$.

Hint about $q=2$: $\quad \pi_1(X)=H_1(X)=0.$

$$\underline{H_2(X)} \underset{\substack{\uparrow \\ \text{see Bott Tu} \\ \text{"path fibration"}}}{\doteq} H_1(\Omega X) = \pi_1(\Omega X) = \underline{\pi_2(X)}$$

$$\underset{\substack{\uparrow \\ \text{abelian}}}{\pi_1(\Omega X)} = \underset{\substack{\uparrow \\ \text{abelian}}}{\pi_2(X)}$$

$$H_1(Z) = \pi_1(Z) / \underbrace{[\pi_1(Z), \pi_1(Z)]}$$

Consequence: $\quad \pi_q(S^n) = \delta^{n,q}\mathbb{Z}$ for $q \leq n$.

(for $q > n$ ... ?)

3.6 Quantum Double Model

[Kitaev 97 ... Hamiltonian lattice gauge theory for G]

Arbitrary cell complex w/ oriented cells Δ

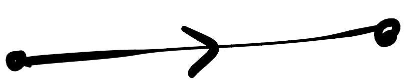
$$\mathcal{H} = \bigotimes_{\substack{\text{1-cells} \\ \Delta_1}} R_{Reg}$$

$$R_{Reg} \equiv \text{span}\{|g\rangle, g \in G\}$$

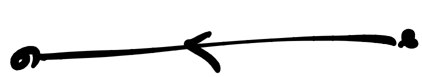
Regular rep. of G ← finite group

$\dim R_{Reg} = |G|$.

$$\left(\begin{aligned} R_{Reg} &= \bigoplus_a (R^a)^{\dim R^a} \\ |G| &= \sum_a (\dim R_a)^2 . \end{aligned} \right)$$



$|g\rangle$



$|\bar{g}\rangle \equiv |g^{-1}\rangle$

$$H = \sum_{v\in\Delta_0}(1-A_v) + \sum_{w\in\Delta_2}(1-B_w)$$

$\boxed{\begin{array}{l}\underline{\text{like } X}: \\ X = \sum_{s=0,1} |s+1\rangle\langle s|\end{array}}$

$$L_+^g \equiv \sum_{h\in G} |gh\rangle\langle h|$$

left multiplication by g.

$$L_-^g \equiv \sum_{h\in G} |h\bar{g}\rangle\langle h|$$

right multiplication.

$\underline{\text{like } Z}$:

$$T_+^g \equiv |g\rangle\langle g|$$

$$T_-^g \equiv |\bar{g}\rangle\langle \bar{g}| = T_+^{g^{-1}}$$

$$\boxed{\bar{g} \equiv g^{-1}}$$

like $XZ = -ZX$

$$L_+^h\, T_+^g = T_+^{hg}\, L_+^h$$

$$L_+^h\, T_+^g = \sum_k |hk\rangle\underbrace{\langle k|g\rangle}_{\delta_{gk}}\langle g| = |hg\rangle\langle g|$$

$L_+^h \, T_+^g$

vs.

$$T_+^k \, L_+^h = \sum_{t\in G} |k\rangle \underbrace{\langle k|ht\rangle}_{\delta_{k,ht}} \langle t| \qquad t = h^{-1}k$$

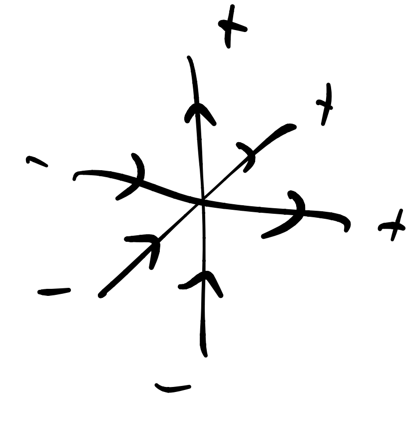
$$= \underline{\underline{|k\rangle\langle h^{-1}k|}} = |hg\rangle\langle g|$$

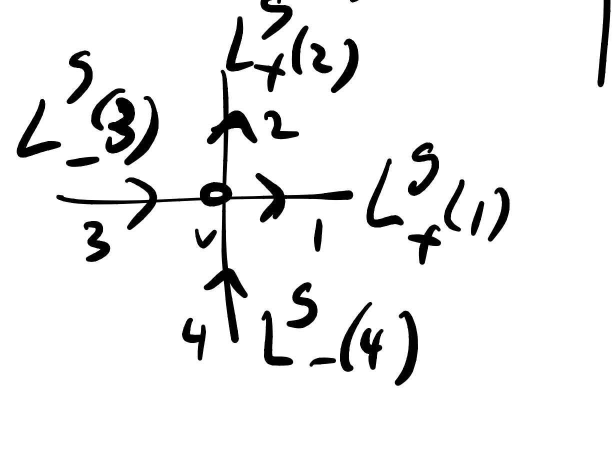
$\underline{k = hg}$.

$$L_-^h \, T_+^g = T_+^{gh^{-1}} L_-^h \qquad \ldots$$

$\pm\pm$.

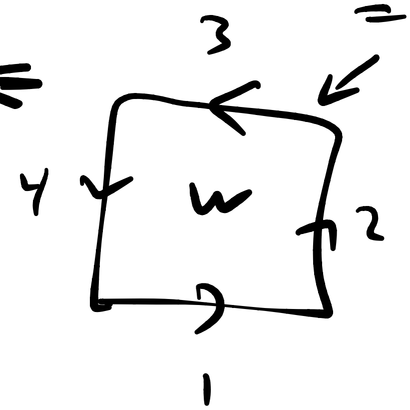
$$A_v = \frac{1}{|G|}\sum_{g\in G} A_v^g \qquad A_v^g = \prod_{l\in\partial^+(v)} L_\pm^{g}(l)$$





$$B_w = \sum_{\{g_1,\dots,g_k\in G \ \text{s.t.}\ g_1\cdots g_k = e\}} \ \prod_{l\in\partial w} T_{\pm}^{g_l}(l)$$

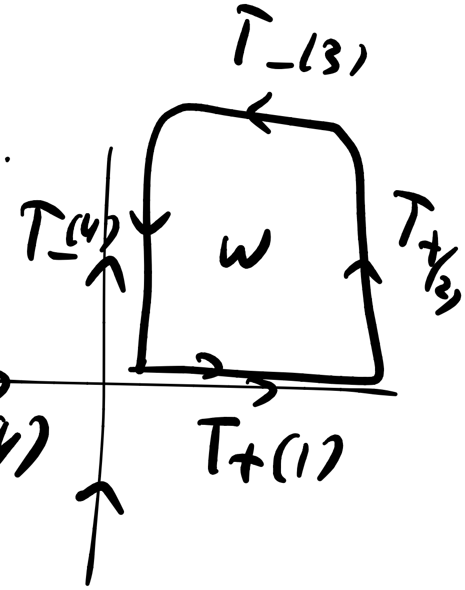
$k = |\partial w|$. $= 4$



$s = \pm$ if

$\partial p = \pm l + \dots$

agrees w/ reference orientation.



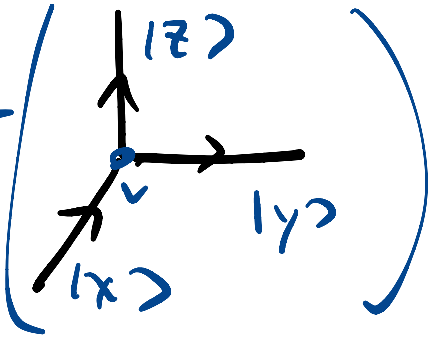
eg $B_w = \sum_{g_1g_2g_3g_4=e} T_+^{g_1}(1)\,T_+^{g_2}(2)\,T_-^{g_3}(3)\,T_-^{g_4}(4)$

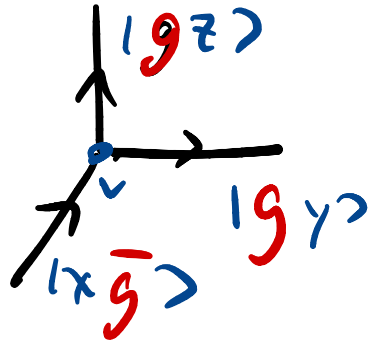
$A_v^2 = A_v$, $B_w^2 = B_w$ projectors

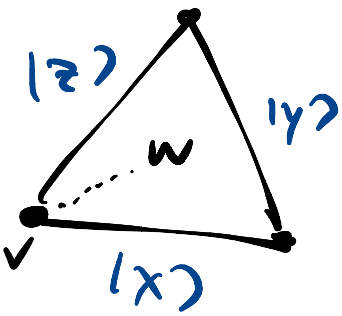
evals 0, 1.

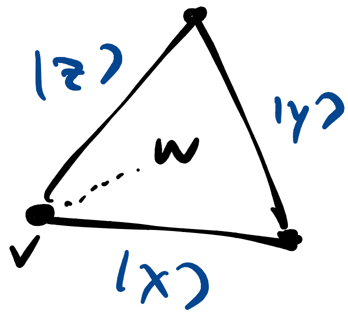
G abelian
$$A \longrightarrow \tfrac{1}{2}(1 + A^{TC})$$
$$B \longrightarrow \tfrac{1}{2}(1 + B^{TC})$$

But, sorry: $A^{TC} = \prod_{+} X$, $B^{TC} = \prod_{\square} Z$.

A_v^g 

$=$ 

$B^h_{s=(v,w)}$ 

$= \delta_{h,xyz}$ 

$$A_v = \frac{1}{|G|}\sum_{g\in G} A_v^g, \qquad B_w = B^e_{s=(v,w)}.$$

H involves

$A_s^g\ B_s^h$ generate the "Drinfeld's quantum double of G"

$$A_s^g A_s^{g'} = A_s^{gg'} \qquad B_s^g B_s^{g'} = \delta_{gg'} B_s^g$$

$$A_s^g B_s^h = B_s^{gh\bar{g}} A_s^g \qquad (A_s^g)^\dagger = A_s^{\bar{g}} \qquad (B_s^g)^\dagger = B_s^{\bar{g}}.$$

claim: $[A_v^g, B_w] = 0 \quad \forall\, v, w.$

$[A_v^g, A_{v'}^{g'}] = 0.$

$[B_w, B_{w'}] = 0$ is obvious

$[T^g, T^{g'}] = 0.$

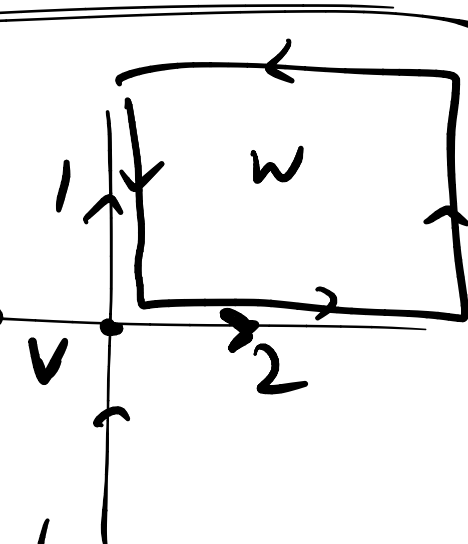
like $[Z, Z] = 0.$

But: while $[X, X] = 0$

$[L_+^g, L_+^h] \neq 0$ if G nonabelian

$[L_+^g, L_-^h] = 0 \checkmark.$

$A_v^h B_w =$



$$\cdots L_+^h(1) L_+^h(2) \sum_{g_1 g_2 g_3 g_4 = e} T_-^{g_1}(1) T_+^{g_2}(2) \cdots$$

$$= \sum_{g_1 g_2 g_3 g_4 = e} T_-^{g_1 \bar{h}}(1)\, T_+^{h g_2}(2) \cdots L_+^h(1) L_+^h(2)$$

$$\begin{cases} \tilde{g}_1 \equiv g_1 \bar{h} \\ \tilde{g}_2 \equiv h g_2 \end{cases}$$

$$= \sum_{\tilde{g}_1 \tilde{g}_2 g_3 g_4 = e} T_-^{\tilde{g}_1}(1)\, T_+^{\tilde{g}_2}(2) \cdots L_+^h(1) L_+^h(2)$$

$$= B_w A_v^h.$$

gs is : $B_w|\Psi\rangle = |\Psi\rangle = A_v|\Psi\rangle$

$\forall\, w, v.$

edge$_e$ $\rightarrow$ group element g_e $\in G$

plaquette w $\rightarrow$ relation $\prod_{e\in\partial w} g_e = e$

$(B_w = 1)$

Elementary Excitations : Analog of e particle: $A_v|\Psi\rangle \neq |\Psi\rangle$

"not invariant under A_v^g"

claim: can find states which are representations of G by A_v^g

$$A_v^g|\Psi_i^a\rangle = \left(D^a(g)\right)_{ij}|\Psi_j^a\rangle$$

$\underbrace{i = 1 \ldots \dim R^a}$

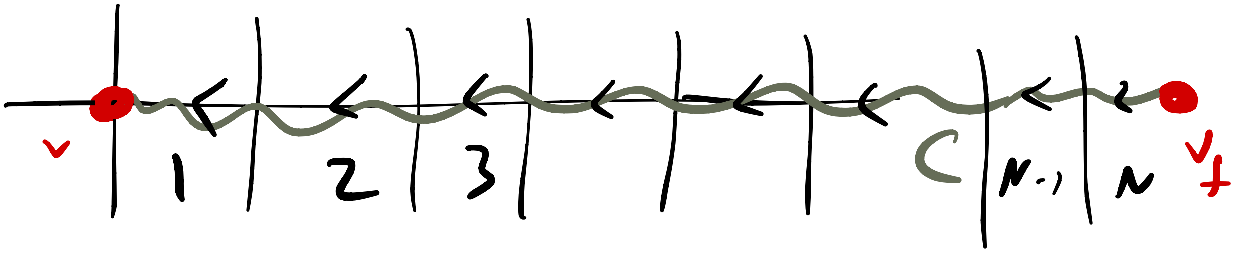
Rep. Matrices:

$$\left(D^a(g)\right)_{ij}\left(D^a(h)\right)_{jk} = \left(D^a(gh)\right)_{ik}$$

$\underbrace{\qquad}_{\text{(Matrix Mult.)}}$

$D^a : G \rightarrow GL(k, \mathbb{R})$

(group homomorphism)



analog of $\underbrace{W_C = \prod_{\ell\in C} Z_\ell}$

what is Z? eg $G = \mathbb{Z}_N$.

$$\boxed{Z = \sum_{g\in \mathbb{Z}_N} D(g)\,|g\rangle\langle g|}$$

ω^k $k = 0..N-1$

$g = \omega^k$.

N.A.

$$\rightsquigarrow \; Z^a_{ij} = \sum_{g\in G} \left(D^a(g)\right)_{ij} |g\rangle\langle g|$$

$i,j = 1..\dim R^a$

a, an irrep. of G.

$$W^a(C)_{if} = Z^a_{ij}(1)\, Z^a_{jk}(2)\, Z^a_{k\ell}(3) \cdots Z^a_{mf}(N)$$

$$= \sum_{g_1\ldots g_N} \sum_{i_2\ldots i_{N-1}} D^a(g_1)_{i i_2}\, D^a(g_2)_{i_2 i_3} \cdots D^a(g_N)_{i_N f}\, |g_1\ldots g_N\rangle\langle g_1 \ldots g_N|$$

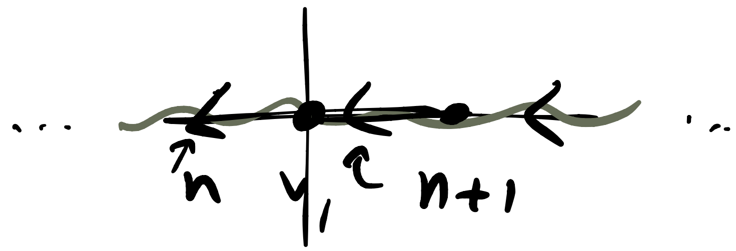
$$= \sum_{g_1 \cdots g_N} D^a(g_1 \cdots g_N)_{if} \, |g_1 \ldots g_N\rangle\langle g_1 \ldots g_N|$$

<u>claim:</u>

$$[A_{v'}, W(C)] = 0 \quad \text{if } v' \neq v, v_f$$

$$[B_w, W(C)] = 0$$

<u>Pf:</u> $A_{v_1} W^a(C)_{if}$



$$= \frac{1}{|G|} \sum_{h \in G} \cdots \underline{L^h_{-}(n)} \, \underline{L^h_{+}(n+1)} \cdots \sum_{g_n g_{n+1}} D^a(g_n g_{n+1})_{i_n i_{n+1}} \, \underline{|g_n g_{n+1}\rangle\langle g_n g_{n+1}|}$$

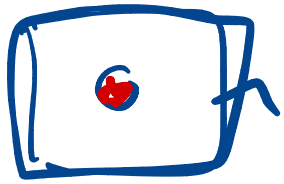
$$= \frac{1}{|G|} \sum_h \sum_{g_n g_{n+1}} \underset{= D^a(\tilde{g}_n \tilde{g}_{n+1})}{D^a(g_n g_{n+1})} \, |\underbrace{g_n \bar{h}}_{\tilde{g}_n}, \underbrace{h g_{n+1}}_{\tilde{g}_{n+1}}\rangle\langle g_n g_{n+1}|$$

$$= W^a(C)_{if} A_{v_1}.$$

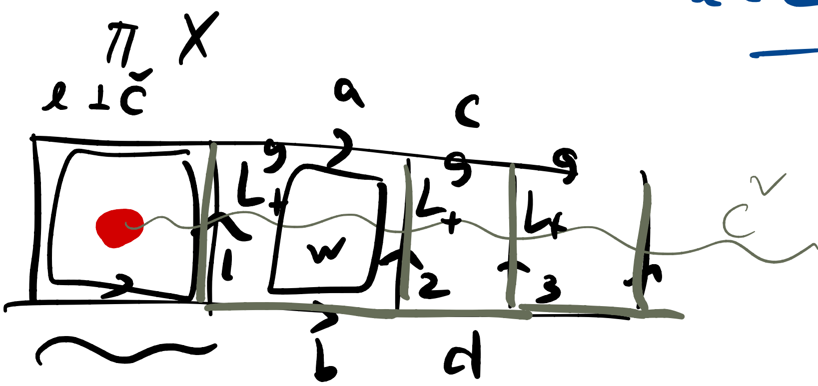
$\boxed{\text{This works in any dimension.}}$

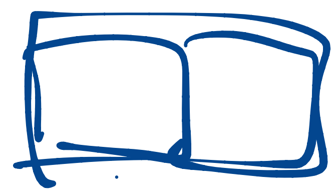
m-excitation is a codim 2 excitation.

a locus AROUND WHICH.



$$\prod_{\ell \in C} g_\ell \neq 1.$$





$\prod g = e$ $\prod S = 1$

$$\bar{g}_1 g_b g_2 \bar{g}_a = e$$

$$\downarrow$$

$$\bar{g}_1 \underline{\bar{g}\, g_b\, g}\, g_2 \bar{g}_a \neq e.$$

$$V_{\check{C}} = L_+^{g}(1) \sum_{g_b} T_+^{g_b}(b)\, L_+^{\bar{g}_b g g_b}(2) \sum_{g_d} T_+^{g_d}(d) \times L_+^{(g_b g_d)^{-1} g (g_b g_d)}(3) \cdots$$

this is special to $d=2$.

Kitaev makes "Ribbon operators" for general excitation IN $d=2$.

$=$ irreps of the Quantum Double.

$= \big(\underbrace{\text{conjugacy class } C_g \text{ of } G}, \text{ IRRep of } Z(g) \big)$

centralizer of g $= \{ h \in G,\ gh = hg \}$ . .

$$\begin{pmatrix} (C_1, \text{irrep of } G) = e \\ (C_g, \text{trivial}) = m. \end{pmatrix}$$

3.7 Fiber bundle & covering maps

$$0 \longrightarrow F \xrightarrow{i} E \xrightarrow{\pi} B \to 0 \quad \text{exact.}$$

induces a long exact seqn. on homotopy ?

$$\to \pi_q(F) \xrightarrow{i_*} \pi_q(E) \xrightarrow{\pi_*} \pi_q(B) \xrightarrow{\partial} \pi_{q-1}(F) \to \cdots$$

$$\cdots \to \pi_1(B) \xrightarrow{i_*} \pi_0(F) \to \pi_0(E) \to \pi_0(B) \to 0.$$

extra requirement: E is the total space of a fiber bundle.

$$\begin{tikzcd} F \arrow[r, hook, "i"] & E \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

$B \equiv$ base. $F \equiv$ fiber.

$F = \pi^{-1}(b)$

① every fiber has a neighborhood

$$\pi^{-1}(U) \cong U \times F$$

homeomorphic

"covering map".

---

② $\forall\, U_\alpha$ in a cover of B: commutes.

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\phi_\alpha"] \arrow[dr, "\pi"'] & U_\alpha \times F \arrow[d, "\text{forget } F"] \\ & U_\alpha \end{tikzcd}$$

$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$

"local trivializations"

A *section* is $s : B \to E$ w/ $\pi \circ s = \mathbb{1}$ on B.

Transition fns : on $\mathcal{U}_{\alpha\beta}$

$$g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \mathcal{U}_{\alpha\beta} \times F \to \mathcal{U}_{\alpha\beta} \times F .$$

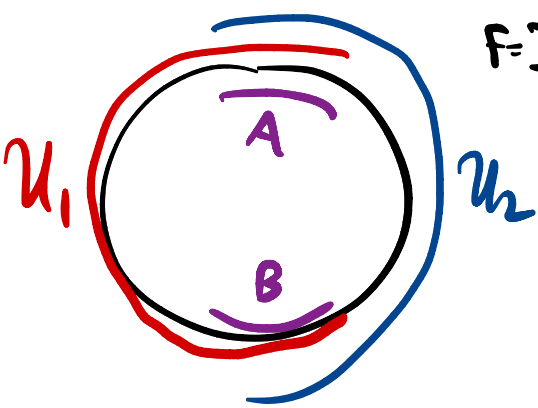
$$\{ g_{\alpha\beta}(x) : F \to F \}_{\substack{x \\ \alpha\beta}} = G \subset \begin{matrix}\text{Homoms:} \\ F \to F .\end{matrix}$$

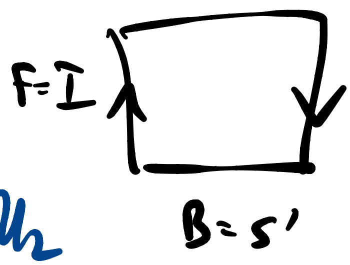
$\uparrow$ structure group of the bundle.

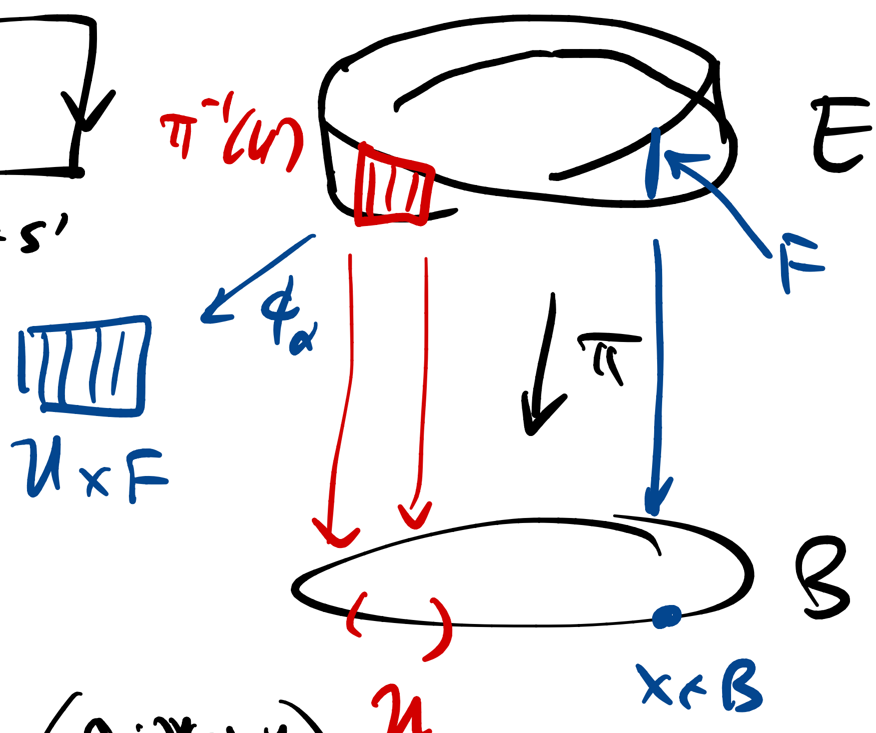
examples: ⓪ $T^2 = S^1 \times S^1$ $\quad \pi : (x,y) \to x .$

$B = S^1$, $F = S^1$. $\quad$ trivial example.

① Mobius band. $B = S^1$, $F = I$.

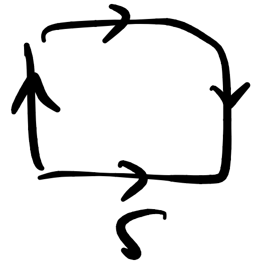






$U_{12} = A \cap B$

$$\phi_{12}(x) = \begin{cases} 1 & \text{if } x \in A \\ g & \text{if } x \in B. \ (g : y \to -y.) \end{cases}$$

eg 1': $K =$ 

S^1 bundle over S^1.

eg 2: **Hopf bundle**. $E =$ unit quaternions $= SU(2) \simeq S^3$.

$\{a + ib + jc + kd\}$

$\supset \{a + ib\}$.

$F = S^1 =$ unit cx #s $\subset$ unit quaternions.

$$S^3 \subset \mathbb{C}^2 = \{(z_0, z_1)\}$$

$$B = \left(S^3 = \{|z_0|^2 + |z_1|^2 = 1\}\right) \Big/ {\scriptstyle (z_0, z_1) \sim (\lambda z_0, \lambda z_1),\ \lambda \in U(1)}$$

$$\begin{array}{ccc} S^1 & \to & S^3 \\ & & \downarrow \pi \\ & & S^2 \end{array}$$

$\leftarrow$ forget overall phase of wave f'n of qubit.

(Bloch sphere.)

$$\pi : S^3 \longrightarrow S^2$$
$$(z_0, z_1) \longrightarrow \underline{\underline{z^\dagger \vec{\sigma} z}}.$$

$$\pi(r_0 e^{i\theta_0}, r_1 e^{i\theta_1}) = \frac{r_0}{r_1} e^{i(\theta_0 - \theta_1)}$$

Fixed $\rho = r_0/r_1$ is a $T^2 \subset S^3$

except at $\rho = 0, \infty$ are 2 linked circles.

$\Rightarrow$

$$\underset{F}{\pi_q(S^1)} \to \underset{E}{\pi_q(S^3)} \to \underset{B}{\pi_q(S^2)} \to \pi_{q-1}(S^1) \to \dots$$

$\underline{\pi_q(S^1) = \mathbb{Z}\,\delta^{q,1}}$ for $q \geq 1$

$\Rightarrow \quad \pi_q(S^3) \cong \pi_q(S^2)$ for $q \geq 3$.

$\Rightarrow \quad \pi_3(S^2) = \mathbb{Z}$.

$= \langle [\pi] \rangle$.

Hopf map.

Universal Cover: $S^1 = \mathbb{R}/\mathbb{Z}$ transl by 2π

General fact: if $X = C/G$ and $\pi_1(C) = 0$

then $\underline{\pi_1(X) = G}$.

$\underline{eg}$: $\pi_1(\Sigma_g) = \pi_1\left({}^{\text{disk}}\!/G \right) \cong G$

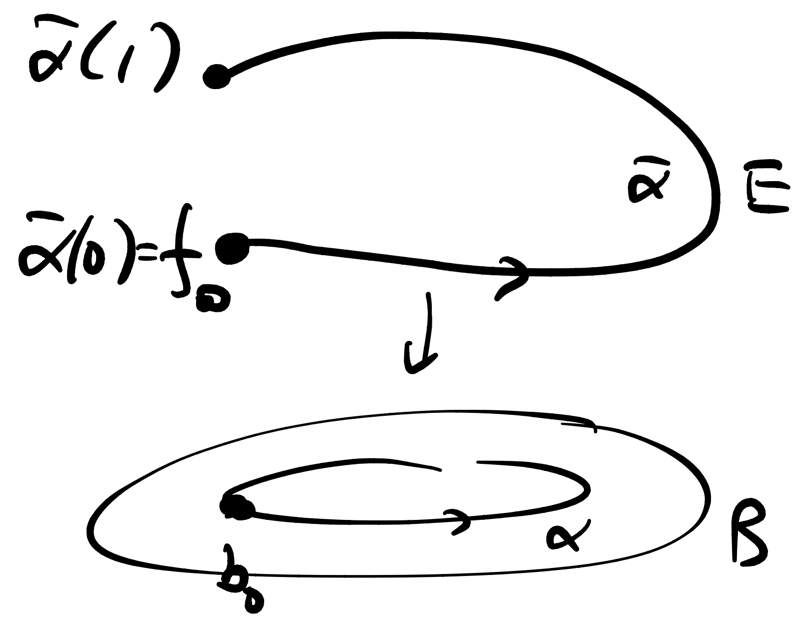
group
g identifications
on ∂(disk)

who is $\partial : \pi_q(B) \to \pi_{q-1}(F)$

$\alpha : (I^q, \partial I^q) \to (B, b_0)$

can $\underline{\text{lifted}}$ to a map $\bar{\alpha} : I^q \to E$.

$\underline{\partial[\alpha] \equiv [\bar{\alpha}(1)]}$



If $\pi : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering map

then $\pi_* : \pi_q(\tilde{X}, \tilde{x}_0) \to \pi_q(X, x_0)$

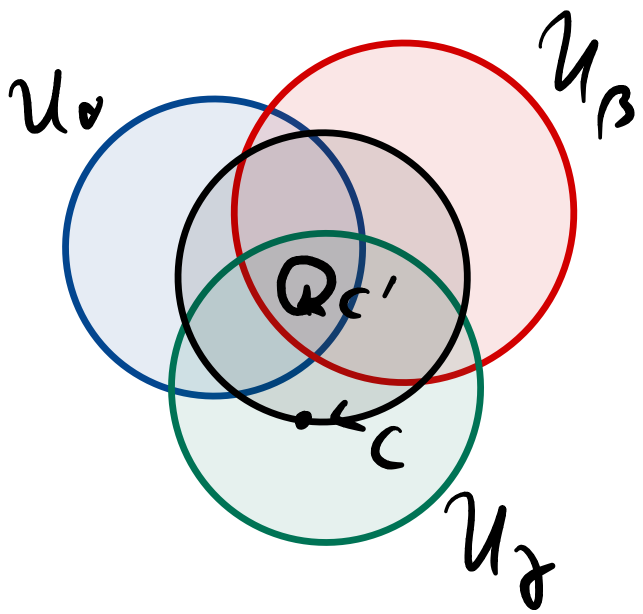
is an isomorphism for $q \geq 2$.

for $g = 1$ the map is injective

$$\pi_1(\tilde{X}) \subset \pi_1(X).$$

Covers of X $\longleftrightarrow$ subgroups of $\pi_1(X)$.

A good cover. $\exists\, U_{\alpha\beta\gamma} \Leftrightarrow C$ is contractible.



$$g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1$$

on $U_{\alpha\beta\gamma}$

cocycle condition

$c \simeq c'$.

$$= (\delta g)_{\alpha\beta\gamma}$$