Last time: line bundles on S^2 $\longleftrightarrow$ $[\,g_{NS} : \text{equator} \to U(1)\,]$
$\simeq S^1$

$\in \pi_1(U(1)) = \underline{\underline{\mathbb{Z}}}$

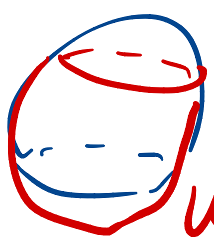
$\longleftrightarrow \quad c_1 = F/2\pi$

$$\oint_{S^2} c_1 = \oint_{eq.} \left(\frac{-i}{2\pi}\, g_{NS}^{-1}\, dg_{NS}\right)$$

$\in \mathbb{Z}$.

---

vector bundle on S^n $\longleftrightarrow$ $[\,g_{NS} : \underset{\simeq S^{n-1}}{\text{equator}} \to G\,]$



$\in \underline{\pi_{n-1}(G)}$.

eg: $\underline{\underline{S^4}}$, $G = SU(2)$ $\qquad \pi_3(SU(2)) = \langle [g^1] \rangle$
$\simeq \mathbb{Z}$.

$$g^1(x) \equiv \frac{x_4 \mathbb{1} + i\,\vec{x}\cdot\vec{\sigma}}{r} \in SU(2) \simeq S^3$$

$x \in S^3(r) \subset \mathbb{R}^4$
$(x_1 \ldots x_4)$
$= (\vec{x}, x_4)$

2x2 matrix
w/ det 1.

$g^\nu(x) = (g^1(x))^\nu \qquad \nu \in \mathbb{Z}$.

$$\int_{S^4} \frac{\mathrm{tr}\, F\wedge F}{16\pi^2} = \left(\underline{\int_{H_N}} + \underline{\int_{H_N}}\right)\underbrace{\left(\frac{\mathrm{tr}\, F\wedge F}{16\pi^2}\right)}_{=\# d\,CS(A)}$$

Chern-Simons 3-form

$$CS(A) \equiv \frac{1}{4\pi}\left(A\wedge dA + \frac{2}{3} A\wedge A\wedge A\right)$$

$$F = dA + A\wedge A$$

$$= \# \oint_{\substack{\text{equator}\\ S^3}} \left(CS(A^N) - CS(A^S)\right)$$

$$A^N = g_{NS}^{-1}(A^S - i\,d)\, g_{NS}$$

$$= \frac{1}{12\pi^2}\oint_{S^3} \underbrace{\mathrm{tr}\; g_{NS}^{-1} dg_{NS} \wedge g_{NS}^{-1} dg_{NS} \wedge g_{NS}^{-1} dg_{NS}}_{=\nu\cdot \text{vol form on } S^3} = \nu \in \mathbb{Z}.$$

winding #.

$g_{NS} = (g')^{\nu}$

OR: for an instanton on $\mathbb{R}^4$ = euclidean config w $F \to 0$ at ∞.

$$\int_{\mathbb{R}^4} \frac{\mathrm{tr}\, F \wedge F}{16\pi^2} = \# \int_{\mathbb{R}^4} d\, CS[A]$$

$$= \# \int_{S^3} CS[A] = \dots \checkmark$$

$A = g^{-1} dg$

$\boxed{\text{instanton \# OR } c_2}$

Fact: $\pi_3(G) = \mathbb{Z}$ for any simple Lie group.

[Milnor, Morse Thy]

Idea: Morse theory on $\Omega G \ni \gamma$

$h \equiv$ length of γ $\qquad \nabla h\big|_\gamma = 0 \iff \gamma$ is a geodesic.

claim: all critical pts on ΩG have even Morse index. $\Longrightarrow H_{odd}(\Omega G, \mathbb{Z}) = 0$ no torsion.

$$\mathbb{Z} \overset{\text{simple}}{=} H_2(\Omega G, \mathbb{Z}) \underset{\text{Hurewicz}}{=} \pi_2(\Omega G) = \pi_3(G).$$

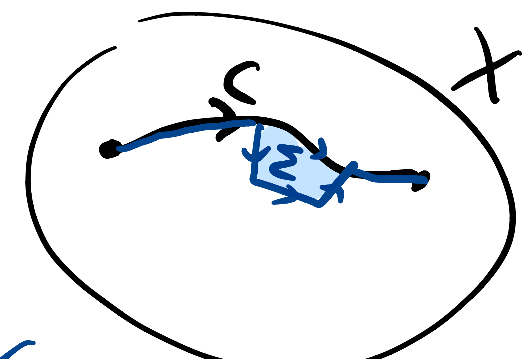
3.8 Quantum Double Model & $\pi_1(X)$.

cell complex Δ triangulate X.

path-connected

A state of QD for G on Δ is $\bigotimes_{e \in \Delta_1} |g_e\rangle$, $g_e \in G$ $\longrightarrow$ G-connection on Δ

$$U(c) = \prod_{e \in c} g_e \in G.$$



Def: A connection is FLAT if $U(c) = U(c + \partial\Sigma) \in G$.

(Σ contractible)

if G continuous

... $\Longleftrightarrow F = 0$

Two connections $\mathcal{U}(C_{xy}) \sim \mathcal{U}'(C_{xy})$

if $\mathcal{U}'(C_{xy}) = \underline{g_x}\, \mathcal{U}(C_{xy})\, \underline{g_y^{-1}}$

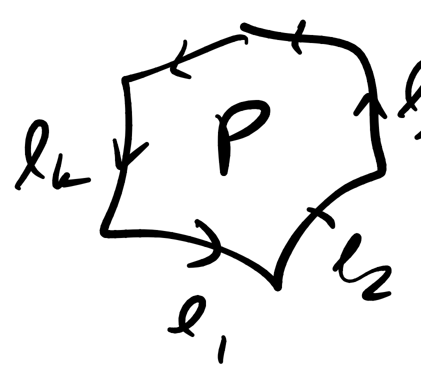
<u>Groundstates</u> of H_{QD} : $\begin{cases} A|\Psi\rangle = |\Psi\rangle \\ B_p|\Psi\rangle = |\Psi\rangle \end{cases}$

$$|\Psi\rangle = \sum_{\{g_l\}_l} \Psi\left[\{g_l\}_l\right] |\{g_l\}_l\rangle$$ general state in $\mathcal{H}_{QD}$

<u>B=1</u> : $\Psi[\{g\}] = 0$ unless

$g_{l_1} \cdots g_{l_k} = e$

if $\partial p = \sum_i^k l_i$.



$\Rightarrow$ <u>FLAT CONNECTION.</u>

$A_s|\Psi\rangle = |\Psi\rangle$ $\forall s$ $\rightarrow$ $\Psi[\{g_{ij}\}_{ij}] = \Psi[\{h_i g_{ij} \bar{h}_j\}]$

$i \to j$ $\qquad A^h_i |\{g\}\rangle$

gs = equal-wt superposition over gauge orbits of connection

$k \to i \to j$ $= |\{h_i g_{ij},\ g_{ki} \bar{h}_i \dots\}\rangle$

i.e. $U(C_{xy}) \to g_x U(C_{xy}) g_y^{-1}$

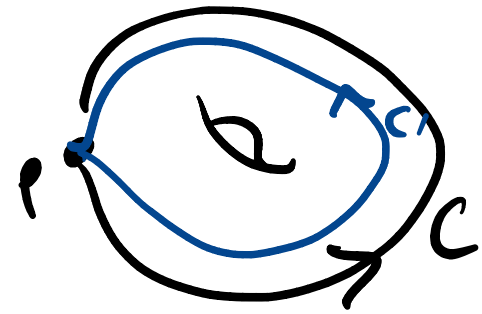
gs $|\Psi\rangle$ of QD on X $\Longleftrightarrow$ flat G-connections on X / gauge equivalence

$\Updownarrow$

Representations of π_1

= group homomorphisms $\rho : \pi_1(X) \to G$ / conjugation in G.

$\boxed{\swarrow}$: given a <u>flat</u> connection on X



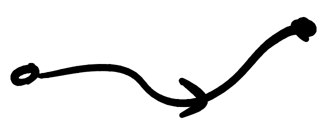
$U(c) \in G$.

$\sim U(c')$

$[c] = [c']$.

$\boxed{\nearrow}$: given $\rho : \pi_1(X) \to G$.

$\rightsquigarrow U(c)$ for <u>closed</u> loops.



?

<u>two steps:</u> let $\tilde{X} \to X$ be the universal cover of X. $\pi_1(\tilde{X}) = 0$.

step 1). Use ρ to make a <u>flat G-bundle</u>

$\equiv$ a bundle w/ a <u>flat atlas.</u>

$\equiv$ transition f'ns are $\in G$ <u>CONSTANTS</u> on $U_{\alpha\beta}$

Step 2: A flat bundle admits a flat connection.

Pf: Flat atlas $\{U_\alpha, \phi_\alpha\}$

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V.$$

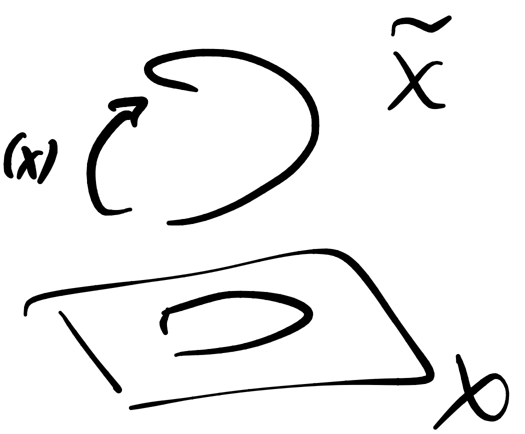
Declare $D^{(\alpha)}_\mu \phi_\alpha = 0.$ i.e $A^{(\alpha)}_\mu \equiv -\phi_\alpha^{-1} \partial_\mu \phi_\alpha$.

$g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ is constant

$0 = \partial_\mu g_{\alpha\beta} \Longrightarrow A^{(\alpha)}_\mu = A^{(\beta)}_\mu$ on $U_{\alpha\beta}$.

$\tilde{X}$ is a principal $\pi_1(X)$-bundle.

w/ constant transition f'ns $g_{\alpha\beta} \in \pi_1(X)$



$E \equiv$ associated bundle

w/ transition f'ns

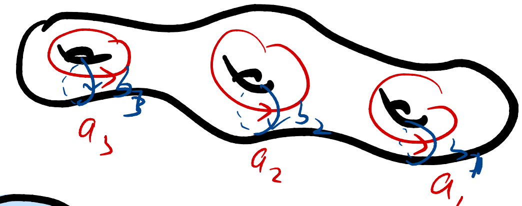
$\rho(g_{\alpha\beta}) \in G$

or $\quad E \equiv (\tilde{X} \times V)/\pi_1(X) \longrightarrow X.$

fiber

$$\left(\begin{array}{l} f(\gamma x) = \rho(\gamma) f(x). \\ f : U \to V \\ \text{local section.} \end{array} \right)$$

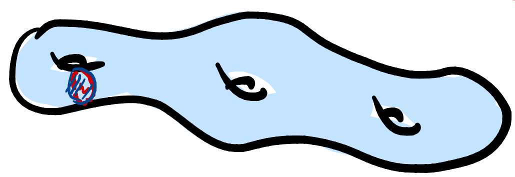
<u>examples</u>: 1) $X_1 = \Sigma_g$



2) $X_2 = B_g$

genus-g handlebody

$\partial B_g = \Sigma_g.$



$\exists p \in B_g$ w $\partial p = b$ $\Longleftrightarrow$ b_i are contractible in B_g.

flat connections on $B_g \subset$ flat connections on Σ_g

$\langle [b_i]_i \rangle \subset \pi_1(\Sigma_g)$

$\Longleftrightarrow$ reps of $\pi_1(\Sigma_g)$ which are trivial on b_i.

→ physical understanding of facts abt $\pi_1(X)$.

---

① subdivision invariance: $\pi_1(\Delta) \stackrel{?}{=} \pi_1(X)$ ✓

from entanglement renormalization of QD:

analogs of CX:

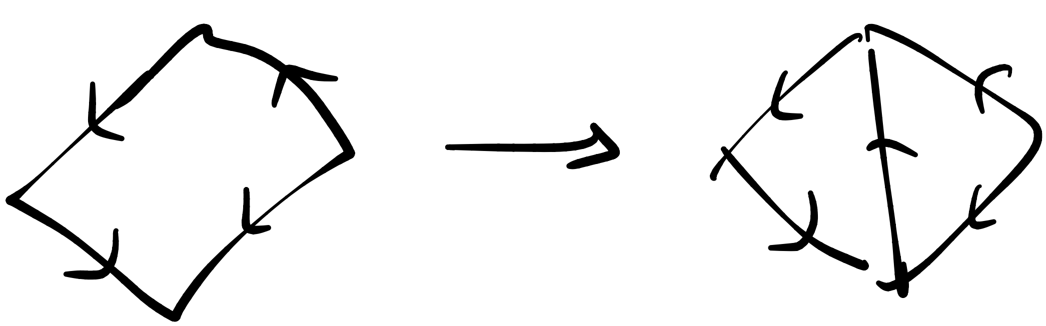
$$CL_{12} \equiv \sum_{g \in G} T_+^g(1) \otimes L_+^g(2)$$

$$CL|g,z\rangle = |g, gz\rangle.$$

$$CR_{12} \equiv \sum_g T_+^g(1) \otimes L_-^g(2) \quad \cdots$$

analogs of $CX(X \otimes 1\!\!1)CX^{-1} = X \otimes X$:

$$CL_{12}\left(L_+^g(1) \otimes 1\!\!1(2)\right) CL_{12}^{-1} = L_+^g(1) \otimes L_+^g(2).$$



② change of coefficients by Higgsing:

$$\Delta H = \Big(\text{terms that make charges condense} \Big)$$ in rep R

Breaks $G \longrightarrow H = \{ g \text{ s.t. } \rho_R(g) = \mathbb{1} \}$.

[take R s.t. H is abelian.]

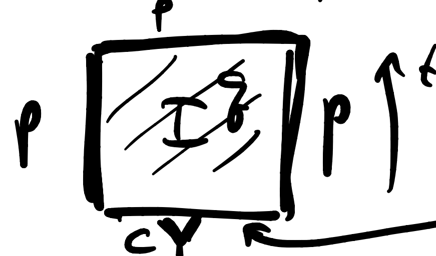
$$\overset{?}{\Longrightarrow} \quad H_1(X) = \pi_1(X) / [\pi_1(X), \pi_1(X)].$$

③ Relative homotopy: Given $Y \subset X$ closed

(base pt $p \in Y$)

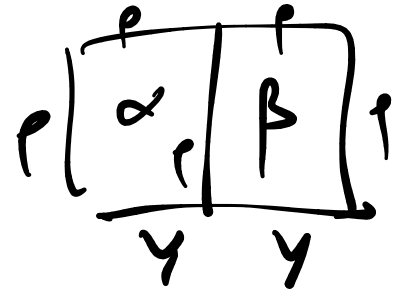
$$\pi_q(X, Y, p) \equiv \pi_{q-1}(\text{paths from } p \text{ to } Y)$$

OR $$= \{ \alpha : (I^q, \partial I^q) \longrightarrow (X, p \text{ or } Y) \} / \simeq$$



$$\Big[\alpha \big|_{t_1 = 0} : (I^{q-1}, \partial I^{q-1}) \longrightarrow (Y, p). \Big]$$

product:



for $q=1$ $\pi_1(X,Y,p)$ is not a group
$= \pi_0(\text{maps} \ldots)$

$$\boxed{\pi_n(X,p,p) = \pi_n(X,p)}$$

Exact sequence on homotopy:

$$\to \pi_k(Y,p) \xrightarrow{i_*} \pi_k(X,p) \xrightarrow{j_*} \pi_k(X,Y,p) \xrightarrow{\partial_*} \pi_{k-1}(Y,p) \to \cdots$$

$$\partial_* : \pi_k(X,Y,p) \to \pi_{k-1}(Y,p)$$

$$\alpha \longmapsto \alpha\big|_{\text{bottom face}}$$

idea: $\Delta H = \left(\text{terms which higgs } G \text{ to nothing in } Y \right)$

$\longrightarrow$ gapped b.c. on ∂Y.

4.1 Topological Field Theory is

A D-dim'l

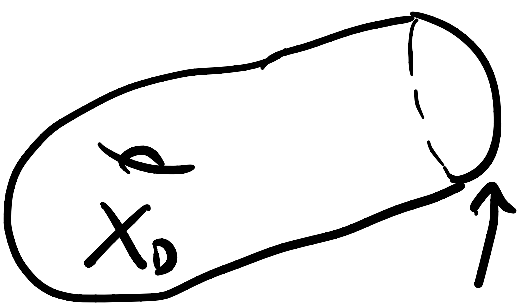
A closed D manifold X $\longrightarrow$ $Z_X = \int_{\text{on } X} D\text{fields}\; e^{i\int_X \mathcal{L}}$; a number $\in \mathbb{C}$. the partition f'n. Z_X

A closed (D-1) manifold Σ $\longrightarrow$ $\mathcal{H}_\Sigma$

A closed (D-2) mf'ld $\longrightarrow$ boundary-condition-changing ops. ; a category

D-3 $\qquad$ 2-category

$\vdots \qquad\qquad \vdots$



$\Sigma_{D-1} = \partial X$

$$\int_{\substack{\text{on } X \\ \text{w bcs on } \Sigma}} D\text{fields}\; e^{i\int_X \mathcal{L}}$$

$$= \Psi[\text{bcs on } \Sigma]$$

$$= \langle \text{bcs on } \Sigma \,|\, \Psi_\Sigma \rangle$$

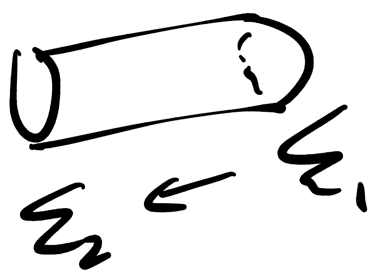
$$|\Psi_\Sigma\rangle \in \mathcal{H}_\Sigma .$$

Axioms :

- $\mathcal{H}_{\emptyset} = \mathbb{C}$
- $\mathcal{H}_{-\Sigma} = \mathcal{H}_{\Sigma}^{*}$

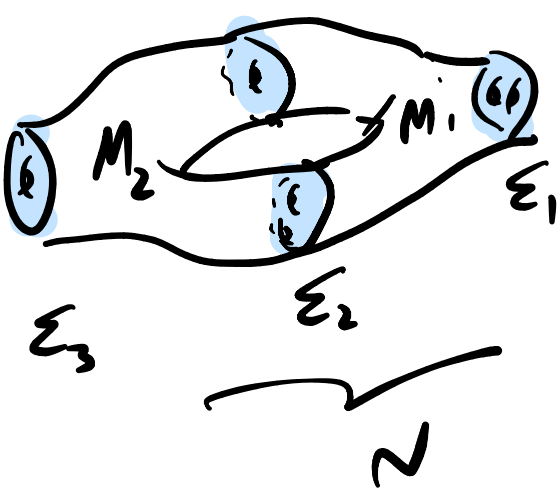
- $\mathcal{H}_{\Sigma_1 \sqcup \Sigma_2} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$

- if $\partial M = (-\Sigma_1) \sqcup (\Sigma_2)$



$$\Rightarrow Z(M) \in \mathcal{H}_{\Sigma_1}^{*} \otimes \mathcal{H}_{\Sigma_2}$$
$$= \mathrm{Hom}(\Sigma_1 \to \Sigma_2).$$

- if $\partial M_1 = -\Sigma_1 \sqcup \Sigma_2$
  $\partial M_2 = -\Sigma_2 \sqcup \Sigma_3$



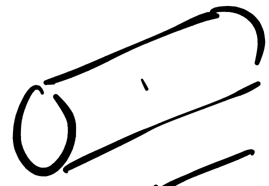
$$N = M_1 \cup_{\Sigma_2} M_2$$

$$Z(N) = Z(M_2) \circ Z(M_1) :$$
$$\mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_3}$$

$$\left(U(t_3, t_1) = U(t_3, t_2)\, U(t_2, t_1). \right)$$

• "**topological**" : [Atiyah]

$$Z(X \times I) : \mathcal{H}_X \to \mathcal{H}_X$$
$$= \mathbb{1}.$$



$$= e^{-HT} \quad \text{ie} \quad \underline{H = 0}.$$

$\Rightarrow$ $\underline{Z(S^1 \times X)}$:

$Z(I \times X) \in \mathcal{H}_X^* \otimes \mathcal{H}_X$

glue X_0 to X_1

$$Z(S^1 \times X) = \mathrm{tr}_{\mathcal{H}_X} \mathbb{1}$$
$$= \dim \mathcal{H}_X .$$

Given a diffeo: $K : X \to X$ on $I \times X$, glue X_0 to X_1 by K $\longrightarrow$ "mapping cylinder" $\underline{S^1 \times_K X}$.

$$Z(S^1 \times_K X) = \mathrm{tr}_{\mathcal{H}_X} \hat{K} \qquad \hat{K} : \mathcal{H}_X \to \mathcal{H}_X$$

Atiyah Publ. Math IHES <u>68</u>
(1989) 175.

(Segal axioms for CFT)

[data in all dimensions = "extended TQFT"

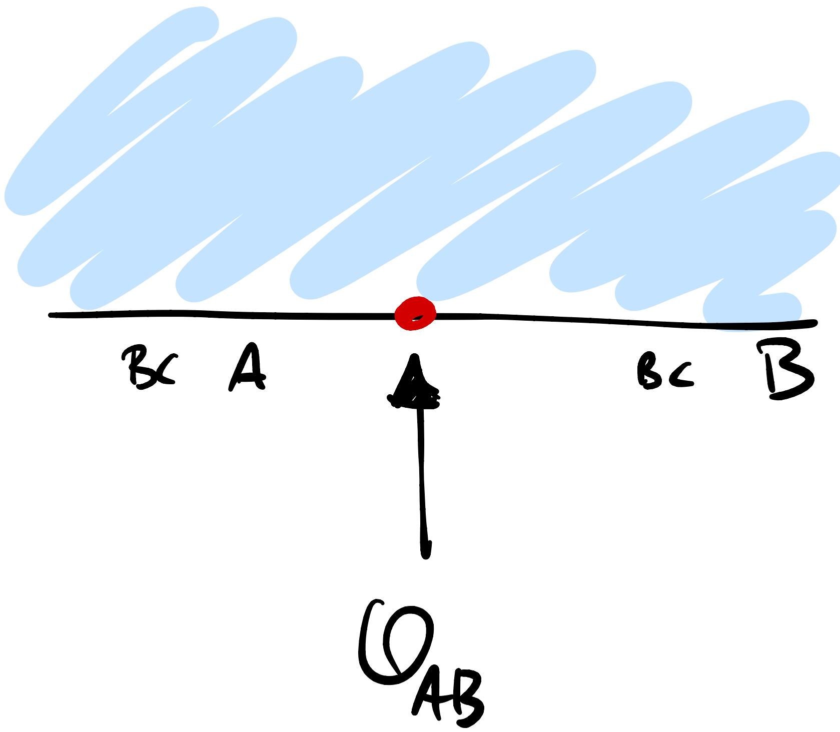
Kapustin ICM 2014]

TQFT : $\underset{\sim}{\text{Bord}} \longrightarrow \text{Vec}$

$\longrightarrow$ spin
oriented
⋮

[Moore lectures on TQFT.]

D=2.



$$\mathrm{obj}(C) = \{\text{Bcs } A, B\}$$

$$\mathrm{Mor}(C)_{A\to B} = \{ \mathcal{O}_{AB} \}$$