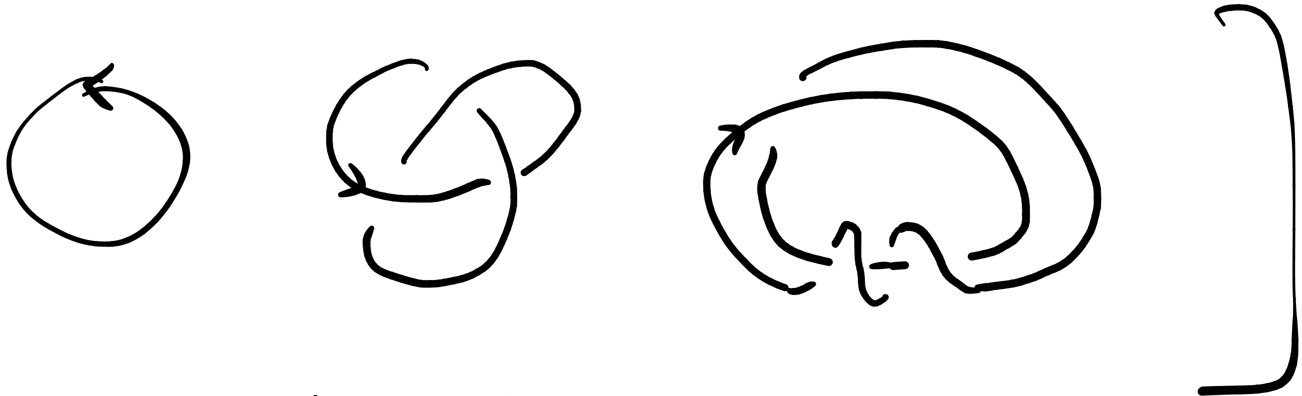
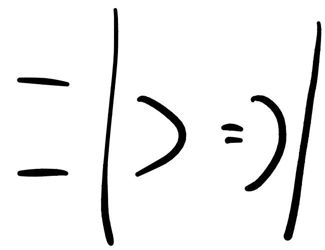
4.2 Chern-Simons Theory & Knot Invts



up to isotopy

eg



Knot $\quad S^1 \to M_3$

link $\quad S^1 \sqcup S^1 \sqcup \dots \sqcup S^1 \to M_3$

Given: oriented smooth closed 3-mfld $M \supset C$

trivial $G = SU(N)$ bundle over M

$E \equiv M \times \mathbb{C}^N$

on E, a connection (a Lie-algebra-valued 1-form on M)

$$A = \underbrace{A_i\, dx^i}_{\tau_{\,i=1..3}} = A_i^A\, dx^i\, \underbrace{T^A}_{N\times N \text{ matrices}}$$

T^A = generators of Lie alg. of G.

λ gauge transf : $A_i \to A_i - D_i \lambda$
infinitesimal

$$\left(D_i \lambda \equiv \partial_i \lambda + [A_i, \lambda] . \right)$$

$$F_{ij} \equiv [D_i, D_j] = \partial_i A_j - \partial_j A_i + [A_i, A_j] .$$

<u>basic object</u> : $Z = \int [DA] e^{i S_{cs}[A]}$

$$S_{cs}[A] \equiv \frac{k}{4\pi} \int_M \operatorname{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

$$= \frac{k}{8\pi} \int_M d^3x \, \epsilon^{ijk} \operatorname{tr} A_i \left(\partial_j A_k - \partial_k A_j + \frac{2}{3} [A_j, A_k] \right)$$

- no metric involved!
- orientation of M was req'd. breaks parity $k \to -k$.
- only 1 derivative vs ~~Maxwell~~ YM has 2 derivatives.
  $\Rightarrow$ more relevant than SYM.
- <u>EOM</u> : $0 = \frac{\delta S_{cs}[A]}{\delta A} \propto F \Rightarrow$ A is a <u>flat connection</u>.

no propagation! <u>Theory</u> of <u>groundstates</u>.

- If $\partial M \neq \phi$ $\quad \delta_\lambda S_{cs}[A] = 0$.

$$A \mapsto A^g \equiv g^{-1} A g - g^{-1} d g$$

$$g : M \to G. \qquad G \text{ simple}$$

eg: $M = S^3$. $\Rightarrow$ classified by $\pi_3(G) = \mathbb{Z}$

"large gauge transfs"

$$\boxed{S_{cs}[A^g] = S_{cs}[A] + 2\pi k \nu}$$

$$\nu = \frac{1}{12\pi} \int_M \mathrm{tr}(g^{-1} d g)^3 \in \mathbb{Z}.$$

$$Z = \int \frac{(DA)}{\mathrm{vol(gauge\ group)}} \underbrace{e^{i S_{cs}[A]}}_{\overset{!}{=} e^{i S_{cs}[A^g]}}$$

$\Longleftrightarrow k \in \mathbb{Z}$.

Note: $k \to \infty$ is weak coupling.

Observables: No (gauge inv't) local operators.

Only Wilson loops: $W_R(C) \equiv \mathrm{tr}_R\, P e^{i\oint_C A}$

for a knot C
a rep R of G.

depends on C only up to isotopy.

$$Z(M, \{C_r, R_r\}) = \int [DA]\, e^{iS_{CS}[A]} \prod_{r=1} W_{R_r}(C_r)$$

$C_r \cap C_{r'} = \phi$.

abelian case

$G = U(1)$.

$$S[a] = \frac{k}{8\pi}\int \epsilon^{ijk} a_i \partial_j a_k \qquad \text{gaussian.}$$

$$Z(S^3, \{C_r, n_r\}) = \int [Da]\, e^{iS} \prod_r e^{i n_r \oint_{C_r} a}$$

$$= \int [Da]\, e^{iS + i\int j a}$$

$$j^i(x) = \sum_r n_r \oint_{C_r} \delta^3(x - x_r)\, dx_r^i$$

$$= \mathcal{N} \exp i \int_x \int_y j_x \underline{D^{-1}_{xy}} j_y$$

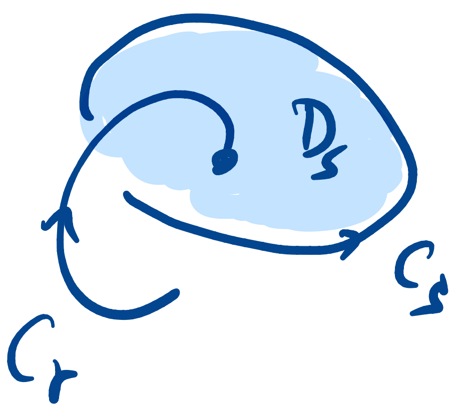
$$S[a] = \iint a_x D_{xy} a_y$$

$$= \mathcal{N} \exp \frac{i}{2k} \sum_{r,s} n_r n_s \underbrace{\oint_{C_r} dx^i \oint_{C_s} dy^j \, \epsilon^{ijk} \frac{(x-y)^k}{|x-y|}}$$

if $C_r \cap C_s \neq \phi$

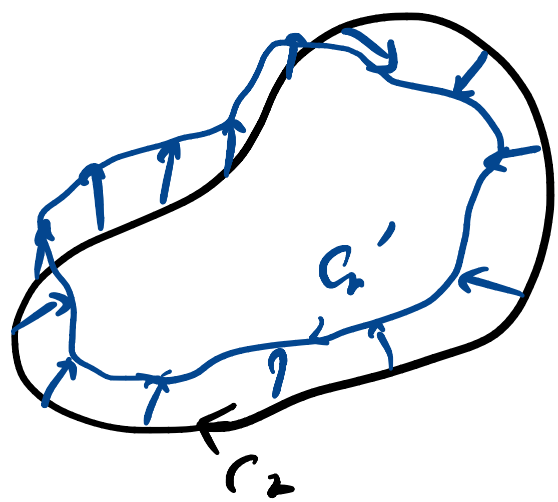
$$\ell(C_r, C_s) = \frac{1}{4\pi} \oint_{C_r} dx^i \oint_{C_s} dy^j \, \epsilon_{ijk} \frac{(x-y)^k}{|x-y|^3} \in \mathbb{Z}$$

Gauss' linking # $\equiv \#(C_r, D_s)$



<u>extra data</u>:

<u>framing of the knot</u>



$$\ell(C_r, C_r) \equiv \ell(C_r, C_r')$$

(point-splitting regularization.)

twist framing by t windings $\ell_{rr} \to \ell_{rr} + t$

$$Z \to Z \underbrace{e^{2\pi i t \frac{n_r^2}{k}}}.$$

Physical ambiguity: particle w/ spin $\frac{n_r^2}{k}$.

must specify not just path, but also its winding.

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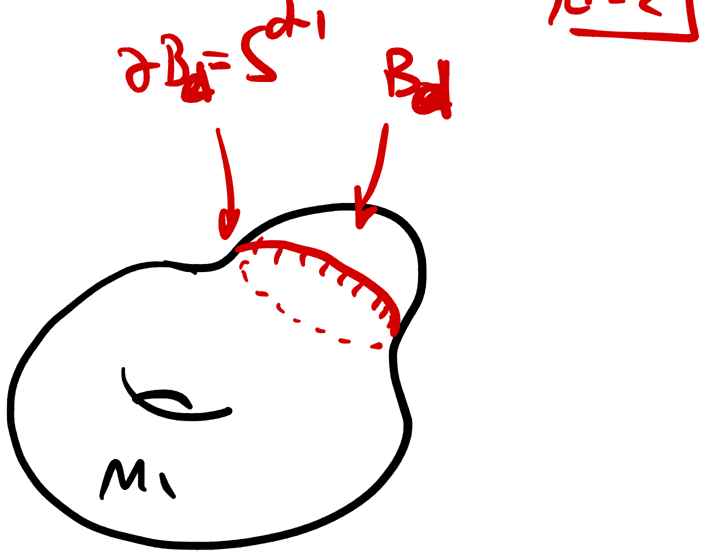
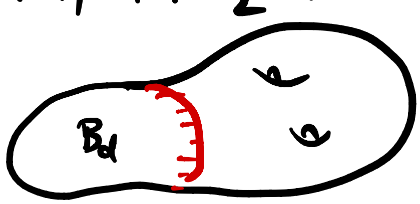
strategy: chop up M. $\boxed{d \geq 2}$

Def: Connected sum.



$M_1 \# M_2$.

Note: 

$$S^d \# M = M$$

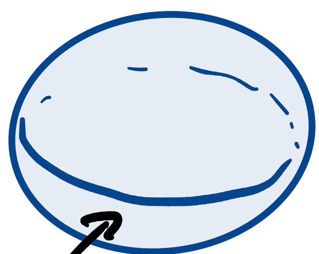
claim: $Z(M_1 \# M_2)\, Z(S^3) = Z(M_1)\, Z(M_2)$

$M = M_1 \# M_2$

aka
$$\frac{Z(M)}{Z(S^3)} = \frac{Z(M_1)}{Z(S^3)} \, \frac{Z(M_2)}{Z(S^3)}$$

(check: $\Rightarrow Z(S^3 \# M) = Z(M)$.)

idea:



$M_1 = B_3$

$M_2 = M \setminus B_3$

In particular $M = S^3 = B_L \# B_R$

B_L

$S^3 \setminus B_L \equiv B_R \approx$ ball.

$$Z(M_1) \in \mathcal{H}_{S^2 = \partial M_1} = |M_1\rangle$$

$$Z(M_2) = \langle M_2 | \in \mathcal{H}^*_{+\partial M_1 = S^2 = -\partial M_2}$$

$$Z(M) = \langle M_2 | M_1 \rangle.$$

$Z(S^3) = \langle B_L | B_R \rangle.$

Fact #1 : $\underline{\mathcal{H}_{S^2} = \mathbb{C}}$. $\left(\begin{array}{c}\text{unique g.s.} \\ \text{on } S^2\end{array}\right)$

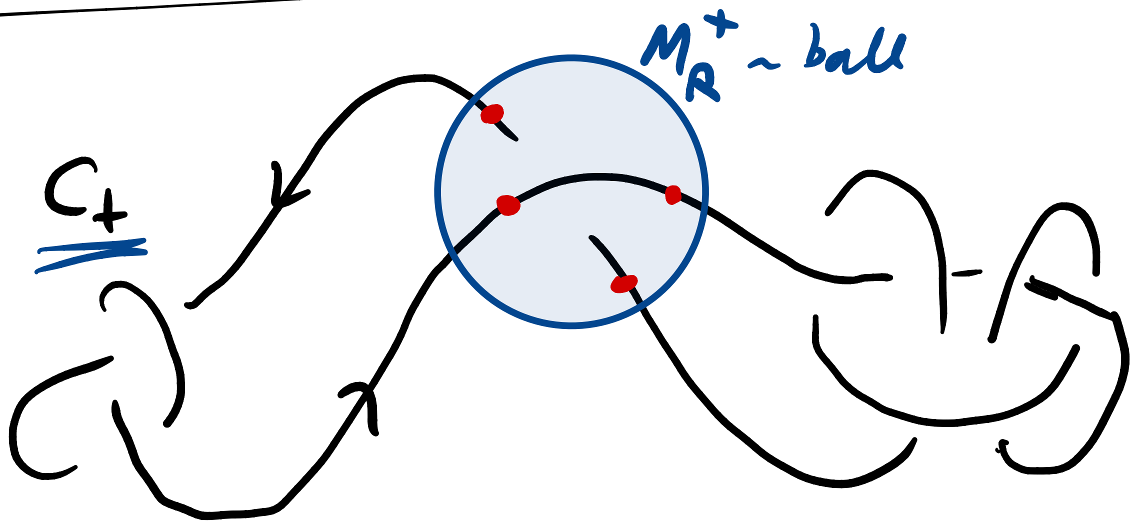
$$\Rightarrow \underset{Z(M)}{\langle M_2 | M_1 \rangle} \underset{Z(S^3)}{\langle B_L | B_R \rangle} = \underset{Z(M_2)}{\langle M_2 | B_R \rangle} \underset{Z(M_1)}{\langle M_1 | B_L \rangle}$$

2

unlinked

$$\Rightarrow \frac{Z(S^3, C_1 \cdots C_s)}{Z(S^3)} = \prod_{r=1}^{s} \frac{Z(S^3, C_r)}{Z(S^3)} .$$

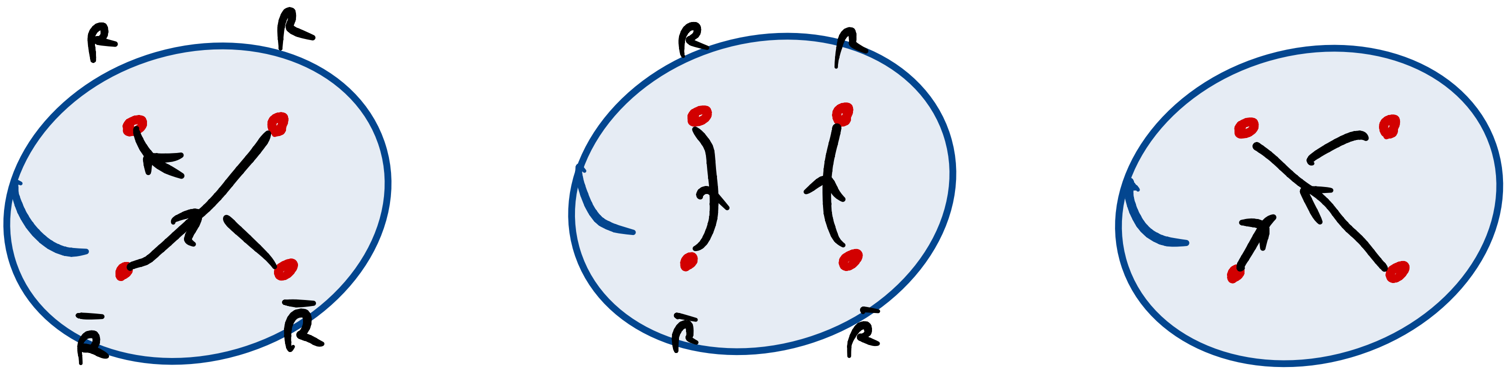
$$\equiv \langle C_1 \ldots C_s \rangle = \prod_{r=1}^{s} \langle C_r \rangle .$$



$\partial M_R^+ = S^2 \setminus \{R, R, \bar{R}, \bar{R}\} \longrightarrow Z(M_R^+ \ldots) = |\Psi^+\rangle \in \mathcal{H}_{S^2 \setminus RR\bar{R}\bar{R}}$

<u>fact 2</u> : for $G = SU(N)$, $R = \square$. N-dim'l rep

$$\mathcal{H}_{S^2,\square\square\bar{\square}\bar{\square}} = \underline{\underline{\mathbb{C}^2}} .$$



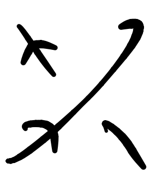
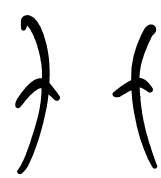
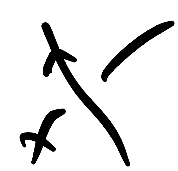
$|M_R^+\rangle$ $\qquad$ $|M_R^0\rangle$ $\qquad$ $|M_R^-\rangle$

$\in \mathcal{H}_{S^2,\square\square\bar{\square}\bar{\square}}$ $\qquad$ $\exists\ \alpha, \beta, \gamma \in \mathbb{C}$

$$\Longrightarrow \underline{\alpha |M_R^+\rangle + \beta |M_R^0\rangle + \gamma |M_R^-\rangle = 0}$$

$\underline{\underline{\langle M_L|}} \in \mathcal{H}^*_{S^2,\square\square\bar{\square}\bar{\square}}$ defined by C^+ in $M - M_R$.

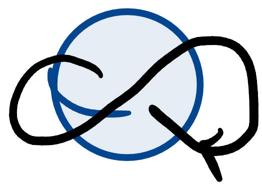
$$\Rightarrow \alpha Z(S^3, C_+) + \beta Z(S^3, C_0) + \gamma Z(S^3, C_-) = 0.$$

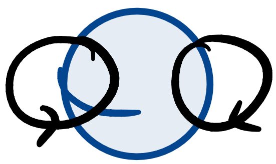
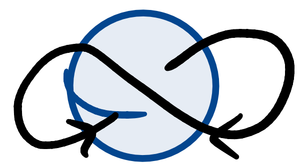
<u>OR</u> : α  $+ \beta$  $+ \gamma$  $= 0$

<u>skein rel'n</u>.

skein rel'n determines $Z(S^3, \text{any knot or link})$

Pf: induction on # of crossings.

eg: $0 = \alpha \langle$  $\rangle$

$+ \beta \langle$  $\rangle + \gamma \langle$  $\rangle$

$$= \alpha Z(C) + \beta Z(C^2) + \gamma Z(C)$$

$$\langle C^2 \rangle = \langle C \rangle^2$$

$$\Longrightarrow \quad \langle C \rangle = - \frac{\alpha + \gamma}{\beta}.$$

Canonical picture: $M = \mathbb{R} \times \Sigma_g$

$\uparrow$ time

choose $A_0 = 0$ gauge.

$$S = \frac{k}{8\pi} \int_{\Sigma_g} \int dt \, \epsilon^{ij} \, \mathrm{tr} \, A_i \frac{d}{dt} A_j \quad + \text{ sources from } W_R.$$

$$[A_x, A_y] = i\delta \quad , \quad \underline{H = 0}.$$

Gauss Law $\underline{\underline{0}} = \frac{\delta S}{\delta A_0} = \frac{k}{4\pi}\epsilon_{ij}F_{ij}^A - \sum_{r=1}^{s} \delta^2(x-p_r)T_r^A$

$\Rightarrow A$ is flat on Σ_g away from sources.

<u>phase space</u> $= \left\{ \begin{array}{c} \text{flat } G\text{-connections} \\ \text{on } \Sigma_g \end{array} \right\}$

<u>finite dim'l.</u>

<u>eg:</u> $\Sigma_g = S^2$. $\quad$ G bundle on S^2

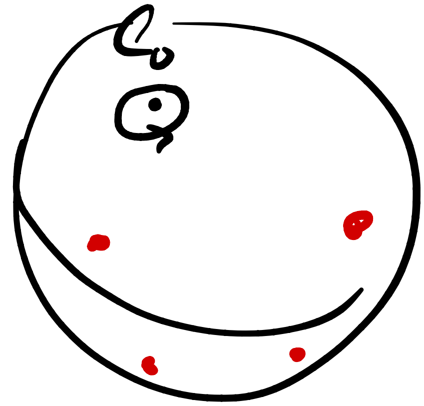
$\Longleftrightarrow \pi_1(G)$

$\Longrightarrow \mathcal{H}_{S^2} = \mathbb{C}$ if $\pi_1(G) = 0.$

S^2 w/ punctures: $\boxed{k \to \infty.}$

$$\mathcal{H} \stackrel{?}{=} \bigotimes_r R_r \equiv \mathcal{H}_0 .$$

$$\underset{\text{flatness}}{0 = \oint_{C_0} A} \;\underset{\text{gauss law}}{=}\; \int_{S^2} \left(\sum \delta \, T^A \frac{4\pi}{k} \right)$$



(total charge on $S^2 = 0$.)

$\mathcal{H} = \underline{G\text{-inv't subspace of } \mathcal{H}_0}.$ ie (singlet)

CLAIM:

$k < \infty$: $\quad R_r \in \{ \text{integrable reps of affine Lie alg } G_k \}$

$\subset \{ \text{irreps of } G \}$

$$\mathcal{H}_{S^2} = \mathbb{C}$$

$$\mathcal{H}_{S^2 \setminus R_a} = \begin{cases} \mathbb{C} & R_a = R_0 \\ 0 & \text{else} \end{cases}$$

trivial id rep

$$\mathcal{H}_{S^2 \setminus \{R_a R_b\}} = \delta_{R_a, R_b^*}\,\mathbb{C} = \delta_{a\bar{b}}\,\mathbb{C}$$

$$\mathcal{H}_{S^2 \setminus \{R_a, R_b, R_c\}} = V^0_{abc}$$

$$\dim V^0_{abc} \equiv N_{abc}$$

$G = SU(N)$:

$$\square \otimes \square = \begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \square\square$$

$$\Rightarrow \quad \underbrace{\square \otimes \square}\otimes \underbrace{\bar{\square} \otimes \bar{\square}} = \left(\begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \square\square\right) \otimes \left(\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \oplus \overline{\square\square}\right)$$

$$= \left(\begin{smallmatrix}\square\\\square\end{smallmatrix} \otimes \overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right) \oplus \left(\square\square \otimes \overline{\square\square}\right) + \text{nonsinglet}$$

$\longrightarrow$ 2 singlets

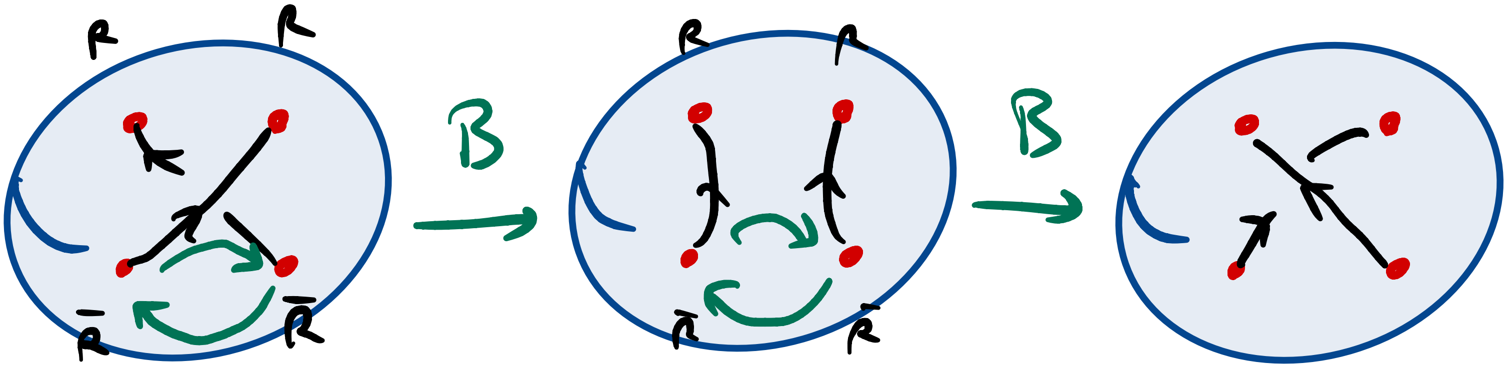
$$\dim \overline{\mathcal{H}_{S^2 \setminus \square\square\bar{\square}\bar{\square}}} \to \mathbb{C}^2 .$$

eg: QD w $G = SL(2, \mathbb{F}_3)$

has 3 2d reps $\underline{2}$

$2 \otimes 2 \otimes \bar{2} \otimes \bar{2} = \underline{1 \oplus 1} + \dots$

BRAIDING:



$|M_R^+\rangle$ $|M_R^0\rangle$ $|M_R^-\rangle$

$|M_R^0\rangle = \hat{B}\, |M_R^+\rangle$

$|M_R^-\rangle = \hat{B}\, |M_R^0\rangle = \hat{B}^2\, |M_R^+\rangle.$

B is a unitary on $\mathbb{C}^2$

$\Rightarrow B^2 - yB - z = 0$ w $y = \mathrm{tr}\, B$, $z = \det B$

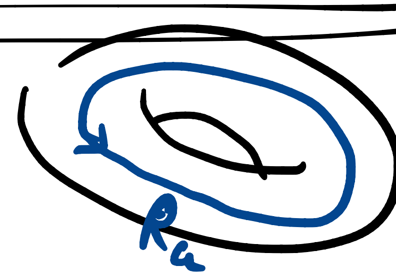
$$\Rightarrow \quad z\langle M_R^{+}\rangle - y\langle M_R^{0}\rangle + \langle M_R^{-}\rangle = 0.$$

$$\begin{cases} \alpha = -q^{N/2} & \beta = q^{1/2} - q^{-1/2} \\ & \gamma = q^{-N/2} \qquad q \equiv e^{\frac{2\pi i}{N+k}}. \end{cases}$$

CFT + framing ambiguity.

$$\langle C \rangle = \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}}.$$

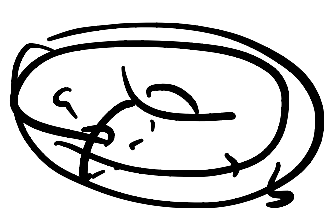
- $\geq 0 \Leftarrow$ reflection positivity.
- $k \to \infty \qquad \langle C \rangle = \mathrm{tr}\, \mathbb{1} = N \quad ✓.$

Dehn surgery $\qquad \mathcal{H}_{T^2} = \mathrm{span}\{$ 

$\}$

$\equiv |V_a\rangle$

on T^2 $SL(2,\mathbb{Z})$ of large diffeos



$$\begin{pmatrix} n \\ m \end{pmatrix} \to K\begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} n \\ m \end{pmatrix}$$

$a, b, c, d \in \mathbb{Z}$

$ad - bc = 1.$

glue M_L = solid torus to $M_R = M \setminus$ solid torus

up to $K \in SL(2,\mathbb{Z})$.

eg: $S^3 = M_L \cup_{T^2} M_R$

M_L

a is contractible in M_L

glue M_L to M_R

by $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\Rightarrow \tilde{M} = \begin{pmatrix} B_2 \text{ glued to } B_2 \\ = S^2 \end{pmatrix} \times S^1$$

$$= S^2 \times S^1.$$

b is contractible

$$Z(M) = \langle M_L | M_R \rangle$$

$$Z'(\tilde{M}) = \langle M_L | \hat{K} | M_R \rangle$$

$\hat{K}$ on $\mathcal{H}_{T^2}$.

$$\hat{K} |v_a\rangle = \sum_b K_a{}^b |v_b\rangle$$

Given $M \supset C$ w R_a

$$Z(M, C) = \langle M_L | V_a \rangle$$

$$Z(\tilde{M}_k, C) = \langle M_L | \hat{k} | V_a \rangle$$

$$= \sum_b k_a{}^b \langle M_L | V_b \rangle$$

$$= \sum_b k_a{}^b \underline{Z(M, R_b)}$$

$$Z(S^2 \times S^1, \{R_a\}) = \dim \mathcal{H}_{S^2 \setminus \{R_a\}}$$

Take $\tilde{M} = S^3$, $M = S^2 \times S^1$, $\underline{k = S}$.

$$Z(S^3) = \sum_b S_0{}^b Z(S^2 \times S^1, R_b) = \underline{S_0{}^0}.$$

$$Z(S^3, R_a) = S_a{}^0 = \langle C_a \rangle Z(S^3)$$

$$\longrightarrow \langle C_a \rangle = \frac{S_a{}^0}{S_0{}^0}.$$

for $G = SU(2)_k$. spin $a/2$, $a = 0 \ldots k$

$$S_{ab} = S_a{}^c \delta_{a\bar{c}} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi(a+1)(b+1)}{k+2} .$$

Verlinde formula.

$$N_{bcd} = \sum_a \frac{S_{ab} S_{ac} (S^{-1})_d{}^a}{S_{a0}}$$

Borel Bott Weil thm :

IRREP of G = $\mathcal{H}$ made from phase space = symmetric space of G.

Dijkgraaf - Witten.