Gapped boundaries and domain walls in $2 + 1d$ topological phases

A. A. Akhtar[1]

[1] *Department of Physics, University of California San Diego, La Jolla, CA 92093, USA*

(Dated: March 18, 2021)

In this paper, we articulate some key ideas pertaining to the rich subject of gapped boundaries and domain walls in $2+1d$ topological phases. We also discuss an alternative perspective in deriving boundary theories motivated by entanglement considerations.

I. INTRODUCTION AND PRELIMINARY

Topological phases and anyon condensation in two spatial dimensions have been explored through the helpful paradigm of Levin-Wen models (LWMs), also called string-net models [1]; the canonical LWM is the $\mathbb{Z}_2$ toric code (TC). These models require an arbitrary unitary tensor category $\mathcal{C}$ as input, and define a lattice whose bulk excitations belong to $Z(\mathcal{C})$, the monoidal center of $\mathcal{C}$. However, gapped boundaries of 2+1d topological phases require additional mathematical structure, first elucidated by Kitaev and Kong in [2]. Though [2] was seminal, more pedagogical treatments are found in [3, 4].

This paper will be structured as follows: in the rest of I, we will introduce the basic categorical language underlying 2+1d topological phases (we assume the reader is familiar with the essential physics); in II, we will summarize the treatment of gapped boundaries through the Frobenius algebra describing the boundary condensate; in III, we will describe about an alternative treatment of boundary theories that follows from considerations about the ground state entanglement data; in IV, we offer some future directions of study.

A *braided, fusion tensor category* is the input for 2+1d topological order. It consists of *simple objects* c_i which describe different species of point excitations ($1, e, m, f$ in TC). Physically relevant categories are *semi-simple*, meaning that every object a is a direct sum of simple objects $a = \oplus_i m_{ai} c_i$, $a, c_i \in C$, $m_{ai} \in \mathbb{Z}_{\geq 0}$.

Morphisms f are structure-preserving maps $f \in \mathrm{Hom}(a, b)$ between objects $a, b \in C$ that describe physical processes. For example, there is only a map between c_i and c_j only if $i = j$. The number of maps depends on the particle type. For example, if c_i is a boson, it has a one-dimensional endomorphism space i.e. $\dim \mathrm{Hom}(c_i, c_i) = 1$. Correspondingly, $\dim \mathrm{Hom}(\oplus_i m_i c_i, \oplus_i n_i c_i) = \sum_i m_i n_i$. In this way, we can label a morphism from c_i to a through basis vectors $c_i \xrightarrow{\alpha} a$ with $\alpha = 1 \ldots \dim \mathrm{Hom}(c_i, a)$. Under conjugation, each anyon is replaced is replaced by its *dual*.

The anyons (simple objects) obey a (commutative, associative, distributive) *fusion algebra* $a \otimes b = \sum_c N_{ab}^c c$, $N_{ab}^c = \dim \mathrm{Hom}(a \otimes b, c)$, and there is a trivial object 1 that represents the vacuum. This can be represented diagrammatically as a fusion vertex labeled by $\mu = 1 \ldots N_{ab}^c$. The eigenvalues of the object N_{ab}^c are known as the *quantum dimensions* and satisfy $d_a d_b = N_{ab}^c d_c$. A loop of anyon a evaluates to d_a. Associativity $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ gives rise to an isomorphism between the associated fusion vector spaces, which in a particular basis is called the F-matrix or *associator*. Consistency then implies the famous pentagon equation. Lastly, we can introduce *braiding* to describe moving a line around another line. This introduces the famous R symbol and hexagon equation. See [3, 4] for a fuller treatment with illustrations!

Note that to describe fermion condensation, one has to consider a *super*-fusion category, which we will not get into for reasons of brevity, but one of the primary differences is the endomorphism space of simple objects is expanded to allow fermion parity odd maps [5].

II. ESSENTIAL CONCEPTS IN THE CATEGORY THEORY OF THE BOUNDARY

The physics of gapped boundaries of 2+1d bosonic, non-chiral topological orders (e.g. the point-like excitations and their fusion data) are determined by the boundary condensate, which is characterized by its commutative, separable, symmetric Frobenius algebra $\mathcal{A}$. For example, in TC, at a rough boundary e particles condense, there is only one point-like excitation $m = f$.

More precisely, an *algebra* $\mathcal{A}$ in a category C is a collection of simple objects $\mathcal{A} = \oplus_i W_{i1} c_i$, $W_{i1} \in \mathbb{Z}_{\geq 0}$, that is equipped with a (trivially) associative product $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and unit morphism $i_{\mathcal{A}}$ and acts trivially on the algebra. In the TC with rough boundary, $\mathcal{A} = 1 \oplus e$. In algebras, the unit 1 appears only once. We can express the product μ in terms of the Hom basis in C, by expanding $\mathcal{A}$ in terms of its simple objects and using the fusion product in C. A *co-algebra*, on the other hand, has a co-product $\Delta : \mathcal{A} \to \mathcal{A} \times \mathcal{A}$ and a co-unit $\epsilon_{\mathcal{A}}$. The co-product describes splitting e.g. in a $SU(2)$ gauge theory, where the simple objects could be irreps of $SU(2)$, splitting could be a process by which a spin S decomposes into two spins $S = S_1 + S_2$. See Fig. 1 for an illustration of these concepts.

The boundary condensate is described by a *Frobenius algebra*, which is an algebra and a co-algebra. Consistency and the various qualifiers imply a number of diagrammatic identities, which can be found in the [2]. The point of all these additional conditions, which happen to have fancy mathematical names, is that the condensate should basically behave like the vacuum at the boundary: you can freely fuse it, braid it, apply F moves, etc.

without changing anything. This is consistent with our discussion of boundary conditions in the toric code, in which the condensed anyon "disappears" at the boundary. Fermions condensates are slightly trickier, because the fermion condensate has non-trivial braiding.

The gapped, boundary excitations (or defects) are then described by the different representations (or modules) of the condensate algebra. Note that for a $1d$ boundary, the category describing the excitations isn't necessarily braided. Consider the rough boundary of the toric code again, where the bulk excitations m, f are indistinguishable at the boundary. Nonetheless, when fusing with the condensate $1 \oplus e$, they transform into eachother. This is what it means for the boundary excitations to form a representation of the condensate algebra. Furthermore, the representations of the condensate algebra $\mathcal{A}$ form a fusion category, and therefore have well-defined fusion rules.

A *module* M of $\mathcal{A}$ in C is also a collection of anyons $M = \oplus_i W_{iM} c_i$. The module M carries maps $M \to C$ and $C \to M$, these are the projection and inclusion maps between M and simple objects in C. They satisfy predictable completeness and orthogonality relations. Furthermore, since M forms a representation of $\mathcal{A}$, there is a linear map $\rho_{\mathcal{A}}^M : \mathcal{A} \times M \to M$ which satisfies a homomorphism property i.e. acting with $\mathcal{A}$ on M twice is like acting with $\mathcal{A}$ on $\mathcal{A}$ and then acting on the product on M. See Fig. 2 for the precise diagrammatic equations. If we act with $\mathcal{A}$ on M from the left (right), M is a left (right) module. As a technical aside, left and right modules are related by twisting. Much of the mathematics here is worked out in [6] in the context of conformal field theory.

Similar to representation theory modules also satisfy a version of the grand orthogonality theorem (see Fig. 2). Furthermore, their matrix elements form a basis. Consequently, any morphism $\phi \in \mathrm{Hom}(\mathcal{A} \otimes j, k)$ can be expressed as a linear combination $\phi = \sum_M \lambda_{M,\{\alpha\}} \rho^{M,\{\alpha\}}$ of linear "representation" matrices labeled by the "irrep" M. A module M may be induced by the product structure μ on $\mathcal{A}$, denoted $\rho_{\mathrm{Ind}\,\mathcal{A}(c_i)}$. This module is in general reducible, and can be used to ascertain the simple (irreducible) modules [3]. Thus, calculating the boundary excitations reduces to the problem of determining the matrix W, whose matrix elements are labeled by simple objects c_i in C and irreducible modules M. Like the character table for finite groups, the rows and columns of this object sum to various dimensions associated to the category (see section 2.3.2 of [3]).

The fusion $\otimes_{\mathcal{A}}$ between boundary excitations M_1, M_2 are determined from the condensate $\mathcal{A}$ and the bulk fusion rules. The new fusion coefficients $n_{xy}^z = \dim \mathrm{Hom}_{\mathcal{A}}(M_x \otimes_{\mathcal{A}} M_y, M_z)$ can be determined by considering the fusion of two bulk anyons with $\mathcal{A}$. What about a gapped interface separating two distinct topological phases? In a sense, this is a special case of what we have already been discussing: if we fold the two dimensional system, the gapped interface becomes a gapped boundary [2]. More prosaically, defects at a junction are understood in terms of *bi-modules* characterized by left, right condensate algebras $\mathcal{A}, \mathcal{B}$, respectively. Its excitations can be ascertained by considering the fusion of $\mathcal{A} \otimes c_i \otimes \mathcal{B}$ that induces a reducible bi-module.

III. AN ALTERNATIVE APPROACH

The approach outline above is based on certain reasonable assumptions about the condensate algebra. Furthermore, it is largely independent of any microscopic details about the system. This makes it sometimes difficult to apply it to physical systems, where it is not a priori clear what, for example, $\mathcal{A}, M$, and W are. An alternative approach to studying gapped boundaries can be found in the entanglement bootstrap [7]. This approach takes as a starting point the conjectured form of the ground state entanglement in $2+1d$ gapped systems. Amazingly, the various categorical underpinnings of topologically ordered systems, e.g. anyons, fusion rules, multiplicities, etc., can be derived solely from two axioms that follow from the entanglement area law [8].

One of they key objects in this approach is the *information convex set* $\Sigma(\Omega, \sigma)$, defined on a reference ground state σ (satisfying the proposed entanglement axioms) and region Ω, which is the set of reduced density matrices ρ on Ω that match σ on balls inside Ω. This set is unchanged under continuous deformations of Ω and its extreme points satisfy a factorization condition. Moreover, the extreme points, which correspond to orthogonal states supported on different subspaces, are of physical significance since they define new superselection sectors (when Ω is an annulus). The fusion information is then derived by considering the information convex set of the two-hole disk.

Extending the analysis to systems separated by a gapped domain wall, this approach finds a new set of superselection sectors, called *parton* sectors. In doing so, the entanglement axioms are relaxed. Parton sectors don't necessarily describe low-energy excitations–though, their composite sectors do include the point-like excitations we usually think of, hence the name parton. They can be localized to either side of the domain wall. Furthermore, they don't fuse in an ordinary way, but rather *quasi*-fuse. The existence of certain parton sectors is determined by measuring the entanglement of certain banana-shaped boundary regions. Interestingly, these regions also allow one to define a notion of topological entanglement entropy for the boundary theory.

IV. CONCLUSION

There are many remaining directions of inquiry, e.g. is there a sense in which the boundary excitations can be braided? Are there topological phases that cannot be separated by a gapped domain wall? Is there a similar framework for discussing gapless boundary defects? How should either of the outlined approaches be modified in the presence of symmetries? What about higher dimensions? etc. We hope this brief article elucidates some of the key ideas regarding gapped boundaries of topological phases and encourages future research in this complex and evolving field.

ACKNOWLEDGMENTS

We acknowledge McGreevy and Physics 239 for explaining Topology and Physics to us. We also acknowledge Bowen Shi for helpful conversations.

[1] M. A. Levin and X.-G. Wen, Physical Review B **71** (2005), 10.1103/physrevb.71.045110.

[2] A. Kitaev and L. Kong, Communications in Mathematical Physics **313**, 351373 (2012).

[3] J. Lou, C. Shen, C. Chen, and L.-Y. Hung, "A (dummy's) guide to working with gapped boundaries via (fermion) condensation," (2020), arXiv:2007.10562 [hep-th].

[4] B. Coecke and E. O. Paquette, "Categories for the practising physicist," (2009), arXiv:0905.3010 [quant-ph].

[5] Z.-C. Gu, Z. Wang, and X.-G. Wen, Physical Review B **91** (2015), 10.1103/physrevb.91.125149.

[6] J. Fuchs, I. Runkel, and C. Schweigert, Nuclear Physics B **646**, 353497 (2002).

[7] B. Shi and I. H. Kim, "Entanglement bootstrap approach for gapped domain walls," (2020), arXiv:2008.11793 [cond-mat.str-el].

[8] B. Shi, K. Kato, and I. H. Kim, Annals of Physics **418**, 168164 (2020).

Appendix A: Appendix

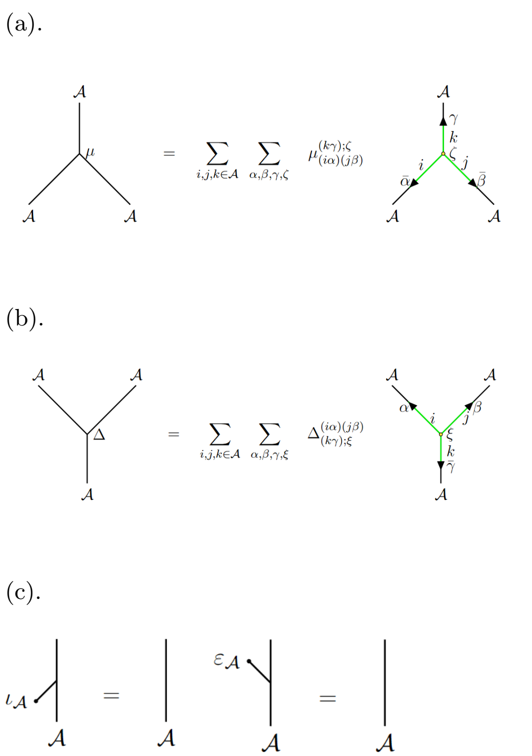

FIG. 1. Some diagrammatic identities on the condensate algebra $\mathcal{A}$ found in [3]. In (a,b), we see the expansion of the (co)product on the (co)algebra in terms of the Hom space in C. In the sum, i, j, k are simple objects, and α, β, γ label the morphism basis from $\mathcal{A}$ to i, j, k respectively. Additionally, ζ labels the fusion channel $\zeta = 1 \ldots \dim \mathrm{Hom}(i \otimes j, k)$. In (c), we see the defining properties of the unit and the co-unit, respectively.

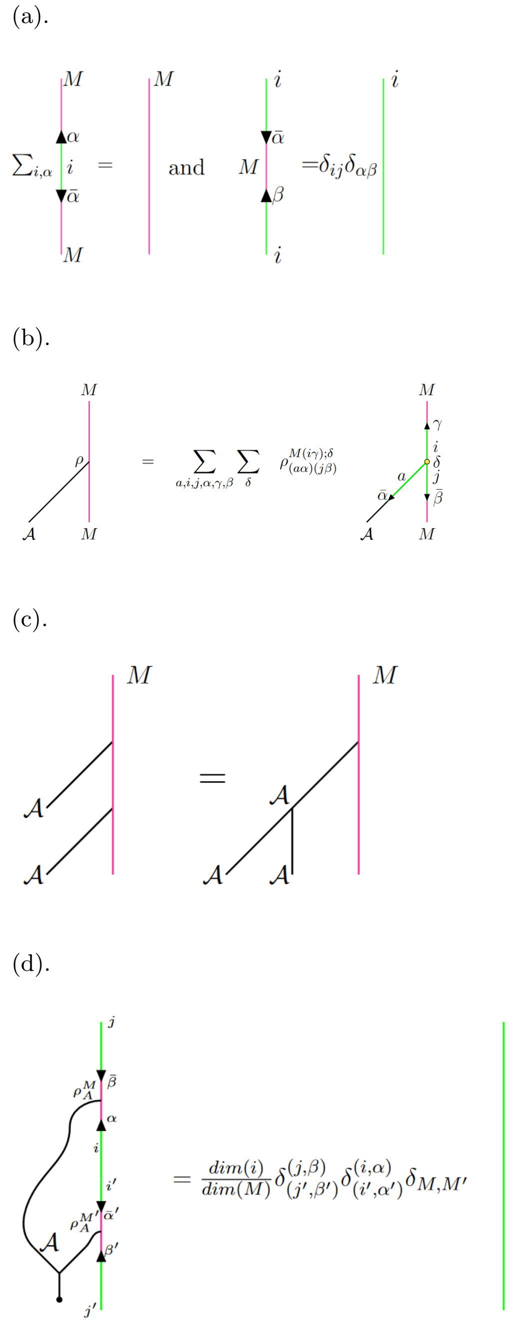

FIG. 2. Some diagrammatic identities on the defect module M found in [3]. In (a), we represent the projection and extension maps between M and simple objects in C. In (b), we show the decomposition of the module into morphisms in C. In (c), we illustrate the homomorphism property of M, which makes it a representation. In (d), we convey the version of the grand orthogonality theorem for modules, showing that different simple modules are orthogonal in a way similar to characters of irreps.