Khonanov Homology

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A short paper explaining the Khonanov Homology, a knot-invariant, the Khonanov Homology is a homology that relates knots of a vector space, and its relation to the Jones Polynomial. This paper will be closely following lectures given by Edward Witten [1] and a mathematical paper from a U Chicago REU [2] and uses figures and equations from both.

JONES POLYNOMIAL

The overarching goal of Khonanov Homology is to create a topological invariant of a knot in $\mathbb{R}^3$. These solutions are found by counting the the solutions to the Jones polynomial. A knot embeds $S^1 \to S^3$. And can be thought of projecting a three dimensional knot into a plane. A link is a mutual assembly of knots in $\mathbb{R}^3$. The projection has a certain integer number of crossings $n \in \mathbb{N}$. When counting the crossings orientation must be maintained and accounted for, thus left handed and right handed crossings have different sign values. Thus the total number of crossings $n = n_+ + n_-$. Convention also defines smoothing, which is when the fibers are stretched and the knot unties as either a 0 smoothing, or 1 smoothing, based on figure 2. The Jones polynomial is then

$$J(q) = \sum_n a_n q^n. \tag{1}$$

FIG. 1: Left and Right handed orientation of crossings

Where a_n is the number of crossings. The standard strategy to count the solutions is to stretch the knot in one direction, for the following case call it the u direction, such that the strands are independent of u except for close to the ends of the knot. Then given a moduli space $\mathcal{M}$ of u-independent solutions. There are solutions

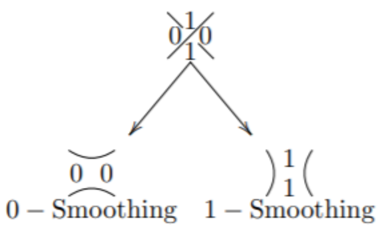

FIG. 2: 0 and 1 Smoothing

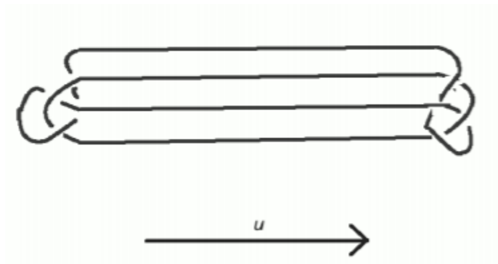

FIG. 3: A knot embedded in $\mathbb{R}^3$ that is limited in range

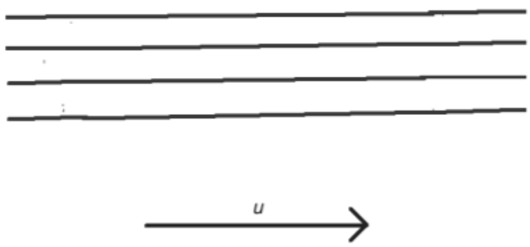

FIG. 4: Infinite parallel strands parameterized by u with $-\infty < u < \infty$

that extend infinitely to $\pm\infty$ denoted by $\mathcal{L}_R$ and $\mathcal{L}_L$ correspondingly. The solutions to a global knot like figure 3 are the solutions in the middle that extend to both ends which is the intersection of $\mathcal{L}_R$ and $\mathcal{L}_L$. This solution is the algebraic intersection number of $\mathcal{L}_R$ and $\mathcal{L}_L$, denoted a_n is the Jones polynomial coefficient. This is effectively

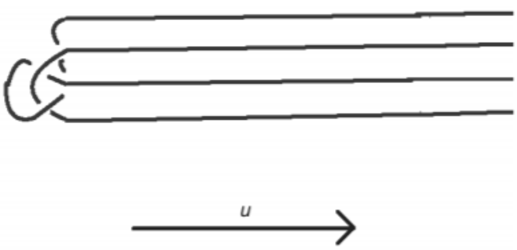

FIG. 5: Parallel strands that go to $u = +\infty$

smoothing out each term in some link L.

$$a_n = \mathcal{L}_L \cap \mathcal{L}_R. \quad (2)$$

KHONANOV HOMOLOGY

Khonanov Homology takes some link diagram from a knot L and transforms it into a bigraded chain complex. By replacing the variable of the Jones Polynomial q with a chain complex,

$$L \rightarrow Khovanov \rightarrow C^{*,*}(L) \rightarrow homology \rightarrow \mathcal{H}(L), \quad (3)$$

the Jones polynomial is being categorified. The Khovanov homology has the following two imporant characteristics

1. The graded Euler characteristic is the unnormalized Jones polynomial
2. If L is related to L' by Reidester moves, then there exists an isomorphism $\mathcal{H}(L) = \mathcal{H}(L')$.

In knot theory, a Reidester move is one of three different local moves. Either a twist/untwist, moving one loop over another loop, or moving a string above or under a crossing. Thus the Khonanov invariant is a knot invariant. In order to define the Khonanov homology, the Kauffman bracket of a link $< L >$ with his own Khonanov bracket $[\![L]\!]$ that follows these axioms.

1. $[\varnothing] = 0 \rightarrow \mathbb{Z} \rightarrow 0$,
2. $[\circ \amalg L] = V \otimes [L]$,
3. Flattening a smoothing of a crossing with a Reidester move reduces to zero.

This procedure allows for the calculation of Khovanov homology

1. Let $\mathbb{Q}[1, x]$, called V, be a graded vector space composed of two basis elements, 1 with degree 1, and x with degree -1. $qdim(V) = q + q^{-1}$.
2. A cube is formed by completely smoothing of each link L where each vertex is associated with a smoothing.
3. Define the cube $0, 1^n$ with each vertex α associated with a group in the chain complex defined by equation 4. r_α is the height function evaluated at α, k_α is the number of circles in the plane of smoothing at vertex α.
4. The chain group at height r_α where $i = -n_- ... n_+$ is the direct sum of all vector spaces at height r_α given in equation 5.

$$V_\alpha := V^{\otimes k_\alpha} r_\alpha + n_+ - 2n_- \quad (4)$$

$$\mathcal{C}(L)^{i,*} := \bigoplus_{r\alpha = i + n_-} V_\alpha. \quad (5)$$

The Khovanov homology is a stronger invariant than the Jones polynomial provides, contains more information specific to the knot of interest, and is a functor. In reference [1], the author demonstrates that the Khonanov homology can be used in string/M theory with the starting point of a six-dimensional superconformal field theory with (0,2) supersymmetry, associated to a simply-laced Lie group G.

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