Supersymmetry, lattice fermions and cohomology theory

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We explore the interplay between Supersymmetry, lattice fermions, cohomology and tic-tac-toe lemma[1]. By exploiting supersymmetry, the number of the ground states for strongly-interacting spinless fermions on different lattices can be computed with the help of tic-tac-toe lemma.

I. INTRODUCTION

The motivation to study supersymmetric lattice fermion model is two-fold. From physics side, the motivation is to study the electrons in the presence of strong repulsive interactions. However, these strongly-interacting systems are notoriously difficult to solve. With commanding the supersymmetry (SUSY) in the lattice fermion model, we can obtain some exact results. These results give us some hints about how these strongly-interacting electrons would behave. From math side, this study can give the result of the cohomology on different lattices or graphs. This relies on the equivalence between computing the number of the ground state in SUSY lattice model and the calculation of the cohomology on the same lattice.

In this paper, I will start from defining the supersymmetric lattice fermion model in Sec. II. Next, the connection with mathematical part, cohomology and tic-tac-toe lemma would be introduced in Sec. III and Sec. IV. Finally, I will go through some applications of how tic-tac-toe lemma helps computing the cohomology in different $1d$ and $2d$ lattices (Sec. V, Sec. VI and Sec. VII).

II. MODEL [2, 3]

In this section, I will start with defining the lattice fermion and how to construct the supercharge operator (the basic property of the supersymmetry theory can be found in Chapter 2 of Professor John Mcgreevy's "Topology from physics" note). The operator $c_i/c_i^\dagger$ annihilates/creates a fermion on site i and they satisfy the anti-commutation relations

$$\{c_i^\dagger, c_i\} = \delta_{ij}, \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0. \tag{1}$$

Next, we can define the supercharge operator in terms of these fermionic operators $c_i, c_i^\dagger$. Note that the supercharge operators need to satisfy the nilpotency condition ($Q^2 = (Q^\dagger)^2 = 0$). A simple choice of the supercharge operator would be $Q = \sum_i c_i^\dagger$. The resultant Hamiltonian would be a trivial constant,

$$H = \{Q^\dagger, Q\} = \sum_{i,j} \delta_{ij} = L. \tag{2}$$

There is no ground state with $E = 0$ in this Hamiltonian. All states come in pairs of opposite fermion parity. A nontrivial choice of the supercharge operator is $Q = \sum_i c_i^\dagger P_i$ where the projection operator P_i can be defined as $P_i \equiv \sum_{j \text{ next to } i}(1 - c_j^\dagger c_j)$. The operation of Q is to add the fermion on site i only when the adjacent sites to site i are empty. The Hamiltonian is

$$H = \sum_{i,j \text{ next to } i} P_i c_i^\dagger c_j P_j + \sum_i P_i. \tag{3}$$

The first term allows the movement of the fermion from i-site to j-site only when the sites adjacent to both of them are empty. The second term can be rewritten in a better form as

$$\sum_i P_i = L - z\sum_i c_i^\dagger c_i + \sum_i V_{\langle i\rangle} \tag{4}$$

where z is the coordination number, $V_{\langle i\rangle} + 1$ is the number of particles adjacent to the site i. When there is no particle around the site i, $V_{\langle i\rangle} = 0$. Therefore, the energy penalty is the same for no particle and one particle around the site i. This is actually an very interesting interaction. To minimize the energy, the second chemical potential term tends to add more particles into the system. However, if there are particles, the penalty from the last term would show up. To compromise, the average separation between two electrons should be 3 sites in the optimal particle numbers. This is called "3-rule"[4]. In fact, we would more focus on the supercharge operator Q itself in the following discussion since the ground state $|\text{gs}\rangle$ with $E = 0$ would satisfy $Q|gs\rangle = Q^\dagger|gs\rangle = 0$.

III. THE GROUND STATE AND COHOMOLOGY[1]

Based on the supercharge operator, it's easy to write down possible particle configurations. To label different particle configurations, we utilize the particle-number symmetry to label each particle configuration. Thus, the full Hilbert space can be decomposed as $\mathcal{H} = \bigoplus_{n=0}\Omega_n$ where Ω_n is the subspace spanned by all possible configurations with n particles. The supercharge operator Q maps from Ω_n to Ω_{n+1} as

$$\Omega_0 \xrightarrow{Q} \Omega_1 \xrightarrow{Q} \Omega_2 \xrightarrow{Q} \cdots \tag{5}$$

Then, we can define the cohomology $H_Q^{(n)}$ of Q as $\text{ket}Q/\text{Im}Q$ within Ω_n. The element in the cohomology $H_Q^{(n)}$ is equivalent with the ground state $|g_n\rangle$ in Ω_n. This

is because the ground state in the supersymmetric lattice model $|g\rangle$ satisfies $Q|g\rangle = Q^\dagger|g\rangle = 0$. This implies that the ground state is in the kernel of Q and can not be expressed as $Q|\psi\rangle$, which is equivalent with the element in $H_Q^{(n)}$. The corresponding Witten index would become $W = \mathrm{Tr}[(-1)^n \mathrm{dim} H_Q^{(n)}]$.

Let's consider a simple example, a periodic chain with 3−sites. I use the binary string to label the particle configuration where 1/0 stands for the occupied/unoccupied site. The full Hilbert space is $\{000\} \bigoplus \{001, 010, 100\}$. The supercharge operator is

$$Q = c_1^\dagger P_{\text{empty } 2,3} + c_2^\dagger P_{\text{empty } 1,3} + c_3^\dagger P_{\text{empty } 1,2}. \tag{6}$$

The detailed calculation is shown in Appendix A. The result shows that there are 2 fermionic ground states. Different boundary condition also affects the ground state degeneracy (also discussed in Appendix A).

IV. 'TIC-TAC-TOE' LEMMA[1]

In Sec. III, we learn how to compute the ground state degeneracy by explicitly constructing all possible particle configurations. However, this method is impossibly applicable to a larger system. The 'tic-tac-toe' lemma can save us.

Let's first decompose the full lattice S into two sublattices S_1 and S_2. We can rewrite the supercharge operator Q as $Q = Q_1 + Q_2$ where $Q_i \equiv \sum_{j\in S_i} c_j^\dagger P_j$. The subspace Ω_n then becomes $\Omega_n = \bigoplus_{p+q=n} K_{p,q}$ where p/q is the number of fermions in the sublattice S_1/S_2. Note that the operations of Q_1 and Q_2 are $Q_1 : K_{p,q} \to K_{p+1,q}, Q_2 : K_{p,q} \to K_{p,q+1}$ (like the diagram in Fig. 1).

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow Q_2 & & \uparrow Q_2 & & \uparrow Q_2 & & \\
K_{0,2} & \xrightarrow{Q_1} & K_{1,2} & \xrightarrow{Q_1} & K_{2,2} & \xrightarrow{Q_1} & \cdots \\
\uparrow Q_2 & & \uparrow Q_2 & & \uparrow Q_2 & & \\
K_{0,1} & \xrightarrow{Q_1} & K_{1,1} & \xrightarrow{Q_1} & K_{2,1} & \xrightarrow{Q_1} & \cdots \\
\uparrow Q_2 & & \uparrow Q_2 & & \uparrow Q_2 & & \\
K_{0,0} & \xrightarrow{Q_1} & K_{1,0} & \xrightarrow{Q_1} & K_{2,0} & \xrightarrow{Q_1} & \cdots
\end{array}$$

FIG. 1. The diagram of the tic-tac-toe procedure[1].

The 'tic-tac-toe' lemma says that the cohomology H_Q is equal to $H_{Q_1}(H_{Q_2}) \equiv H_{12}$ **if all nontrivial elements in H_{12} have the same particle numbers on S_2**. If the sufficient condition is not satisfied, further work is required to connect H_{12} with H_Q.

Let's show the power of 'tic-tac-toe' lemma in a $1d$ periodic chain with 9-sites. I choose sites $3, 6, 9$ in S_2 and the rest of the sites in S_1. To compute H_{12}, we start with finding H_{Q_2}. Since each site in S_2 is not adjacent to each other, we can consider them independently. This is also the reason why this setup is used. Take the site 3 as an example. There are two possibilities, occupied or unoccupied. If the site 3 is occupied, then this configuration $|\psi_1\rangle$ is exact $|\psi_1\rangle = Q_2|\phi\rangle$ and is excluded in H_{Q_2}. If the site 3 is unoccupied with empty adjacent sites, this configuration $|\psi_2\rangle$ would not be close since $Q_2|\psi_2\rangle \neq 0$. The only nontrivial possibility is that the site 3 is unoccupied due to the filled adjacent site. Therefore, either site 2 or site 4 is filled. This result can be applied onto each site in S_2 independently. Here, one might guess there would be total $2^3 = 8$ configurations. However, we also need to make the particles in S_1 not adjacent to each other. If we start from filled site 4, the sites 7 and 1 would also be the filled sites. There are only two different configurations in H_{Q_2}, $|100100100\rangle$ and $|010010010\rangle$. They are both close under Q_1 and not exact since there is no element $|\phi\rangle$ in H_{Q_2} such that $|100100100\rangle = Q_1|\phi\rangle$ or $|010010010\rangle = Q_1|\phi\rangle$. We can conclude there are two nontrivial cohomology classes in H_{12} for 3 particles. The cohomology for all the other particle numbers is trivial. Since the particle number in S_2 is the same for these two configurations, there are also two nontrivial cohomology classes for H_Q. Similarly, this choice of S_1, S_2 can be used to compute the cohomology class for all the open/periodic chain with arbitrary lengthes. Their results are as follows:

The cohomology of Q on the periodic chain with L sites has:

- 2 non-trivial cohomology classes with j fermions if $L = 3j$
- 1 non-trivial cohomology classes with j fermions if $L = 3j + 1/3j - 1$

The cohomology of Q on the open chain with L sites has:

- 1 non-trivial cohomology classes with j fermions if $L = 3j/3j - 1$
- 0 non-trivial cohomology classes with j fermions if $L = 3j + 1$

V. A SIMPLE $2d$ EXAMPLE: MARTINI LATTICE[4]

Even though the tic-tac-toe lemma can save us in the previous $1d$ chain, one might wonder whether this simplicity can be generalized into higher dimension. In this section, I will introduce a simple $2d$ Martini lattice and show that the construction of the nontrivial cohomology classes is similar with $1d$ chain. Eventually, the number of the ground states is equal to the number of dimer covering of the honeycomb lattices.

The Martini lattice is formed by replacing every other site in a hexagonal lattice with a triangle. S_1 is chosen as the sites in the triangles. The rest of the sites

are in S_2. Note that each site in S_2 is not adjacent to each other again. Let's construct the possible configurations in H_{Q_2}. Based on our previous experience, our ultimate goal is to make every sites in S_2 unoccupied non-trivially. This turns out to make one of the adjacent sites in S_1 filled. One of the possible particle configurations is shown in Fig. 2. Constructing the ground state in the Martini lattice is easy but counting the total number is not simple. One way is to define a dimer for each site in S_1 and the orientation is determined by the corresponding filled site in S_2. The number of the ground states turns into the number of dimer covering of the honeycomb lattices[5]. In fact, the entropy of the ground state ($S \equiv \log(\text{number of ground states})$) increases linearly with the system size which suggests that the system is "super-frustrated"[4].

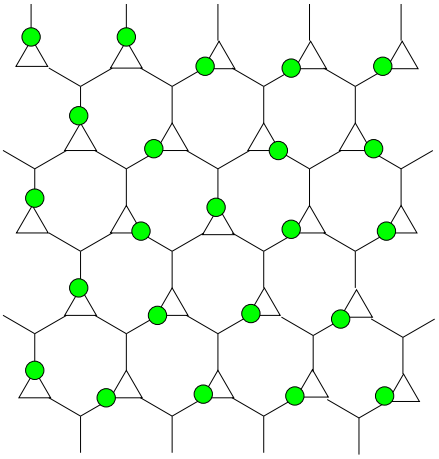

FIG. 2. One of the possible particle configurations of the ground state in Martini lattice[4].

VI. $2d$ SQUARE LATTICE: TILTED RECTANGLES[1]

After showing a special $2d$ example, I will get back to the familiar $2d$ example, square lattice. Let's focus on a special example on the square lattice: tilted rectangle. The idea of defining the tilted rectangular is to decompose a $2d$ plane in terms of a bunch of $1d$ chains. The whole cohomology problem in the tilted rectangle becomes the cohomology of many independent $1d$ chains.

The formal definition of the tilted rectangle $\mathcal{R}(M,N)$ is as the points (x,y) which satisfy $y \le x \le y+M-1$ and $-y+1 \le x \le -y+N$. The example of $M=N=6$ is shown in Fig. 3. The central enclosed green square is the tilted rectangle. The orange points are in S_1 The points connected with blue curves are in S_2. The points in S_1 are now not adjacent to each other which means that we can independently put the electrons on the sites in S_1.

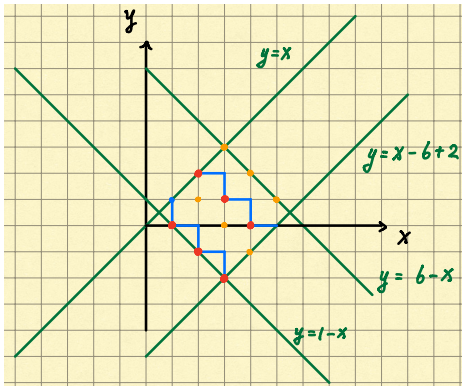

FIG. 3. The tilted rectangle for $M=N=6$.

Again, let's start with computing H_{Q_2}. Since the sites in S_2 are independent chains, we can check individual chains separately. My starting point is the left-lower blue curve. When the corresponding right-upper orange points are filled, they will lead to 3 free and unoccupied sites in the left-lower blue curve (red points). These free and unoccupied sites would not be in H_{Q_2}. Thus, all these three orange points should be unoccupied. By performing this analysis for each chain, we can conclude all sites in S_1 are unoccupied. Each chain in S_2 now becomes an independent open chain and the previous result of the open chain can be directly used here. The length of each open chain is M. The cohomology of the tilted rectangle is

- 1 non-trivial cohomology classes with j fermions if $M = 3j/3j-1$
- 0 non-trivial cohomology classes with j fermions if $M = 3j+1$.

VII. 2D SQUARE LATTICE ON THE TORUS

The simplicity in the tiled rectangles relies on those isolated sites in each chain. If we apply the periodic boundary condition on the end points of the chain, the cohomology can be still easily computed from the previous result of the periodic chain. However, if the square lattice is doubly periodic, there are no such isolated sites and the same trick can not be applied. For doubly periodic lattice, some sites in S_1 would be also filled. The detailed discussion of finding $H_{12} = H_{Q_1}(H_{Q_2})$ can be found in[1]. The basic idea is to start from filling sites in S_1. Then, some sites in S_2 become forbidden to be occupied and those free sites would form open chain. The full chain consists of many open segments. Since we know the cohomology of the open segment with $3j+1$ sites is trivial, this leads to a constraint for the particle configurations in S_1. These configurations would form H_{Q_2}. Finding H_{12} requires the discussion of the elements in

H_{Q_2} with different particle number in S_2. Since the configurations now have different particle number in S_2, tic-tac-toe lemma is not guaranteed to work. Therefore, the last work is to construct H_Q from the elements in H_{12}. Basically, we need to apply Q onto the element $|\phi\rangle$ in H_{12}, $Q|\phi\rangle = Q_1|\phi\rangle$. The operation of Q_2 is guaranteed since this element is chosen from H_{Q_2}. However, the operation of Q_1 is not necessarily to vanish since the subspace is not limited in H_{Q_2}. By expressing $Q|\phi\rangle = Q_2|\psi\rangle$, we can define a new state $|\phi_1\rangle = |\phi\rangle - |\psi\rangle$. If $Q|\phi_1\rangle = 0$, the state $|\phi_1\rangle$ is in the kernel of Q. However, if $Q|\phi_1\rangle$ belongs to another element $|\tilde{\phi}\rangle$ in H_{12}, neither of $|\phi\rangle$ nor $|\tilde{\phi}\rangle$ are not in the kernel of H_Q. With this procedure, we can build the connection between H_Q and H_{12}.

VIII. SUMMARY

In this paper, we started from the definition of the SUSY lattice fermion model and built the connection between the number of the ground state and the cohomology. Furthermore, we utilize the tic-tac-toe lemma to help solving the cohomology on different $1d$ and $2d$ lattices. An interesting direction to explore in the future would be whether the same method can be applied to the spinful fermions.

ACKNOWLEDGMENTS

I greatly thank Professor John McGreevy for teaching us the topology from Physics course and exposing us to various deep and interesting connections between the topology terminology and the models in physics.

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Appendix A: Cohomology calculation and the ground state in SUSY Lattice Fermion Model

For a periodic chain, one can have $C_0 = \{000\}, C_1 = \{100, 010, 001\}, C_2 = \{0\}$. For clarity, let's write down the supercharge operators explicitly,

$$\begin{aligned} Q &= c_1^\dagger P_{\text{empty } 2,3} + c_2^\dagger P_{\text{empty } 1,3} + c_3^\dagger P_{\text{empty } 1,2} \\ Q^\dagger &= c_1 P_{\text{empty } 2,3} + c_2 P_{\text{empty } 1,3} + c_3 P_{\text{empty } 1,2}. \end{aligned} \tag{A1}$$

Let's explicitly compute cohomology of Q on each n particle subspace C_n. For C_0, the image of Q is $\{\}$ since there is no C_{-1}. The kernel is also $\{\}$ since $Q|000\rangle \neq 0$. Thus, $H_Q^{(0)} = 0$. For $n = 1$, the image of Q is $|100\rangle + |010\rangle + |001\rangle$ which suggests the equivalence relation $|001\rangle = -|100\rangle - |010\rangle$. The kernel of Q is $\{100, 010, 001\}$ since $Q|100\rangle = Q|010\rangle = Q|001\rangle = 0$. Thus, we have 2 nontrivial cohomology class $H_Q^{(1)} = \{100, 010\}$. For $n = 2$,$H_Q^{(2)} = 0$ since the kernel and the image of Q are both zero.

For an open chain, one can have $C_0 = \{000\}, C_1 = \{100, 010, 001\}, C_2 = \{101\}, C_3 = \{0\}$. The supercharge operators are a little bit different from the periodic chain.

$$\begin{aligned} Q &= c_1^\dagger P_{\text{empty } 2} + c_2^\dagger P_{\text{empty } 1,3} + c_3^\dagger P_{\text{empty } 2} \\ Q^\dagger &= c_1 P_{\text{empty } 2} + c_2 P_{\text{empty } 1,3} + c_3 P_{\text{empty } 2}. \end{aligned} \tag{A2}$$

Again, we can compute cohomology of Q on each n particle subspace C_n. For C_0, the image of Q is $\{\}$ since there is no C_{-1}. The kernel is also $\{\}$ since $Q|000\rangle \neq 0$. Thus, $H_Q^{(0)} = 0$. For $n = 1$, the image of Q is $|100\rangle + |010\rangle + |001\rangle$ which suggests the equivalence relation $|001\rangle = -|100\rangle - |010\rangle$. The kernel of Q is $\{100, 010\}$ since $Q(a|100\rangle + b|010\rangle + c|001\rangle) = (a + c)|101\rangle \Rightarrow a = -c$. Thus, the independent coefficients are a, b which suggests the independent basis is $\{100, 010\}$. Thus, we only have 1 nontrivial cohomology class $H^{(1)}$. For $n = 2$, $H_Q^{(2)} = 0$ since the kernel is zero.