Characteristic (anomaly) polynomials for spacetime symmetries

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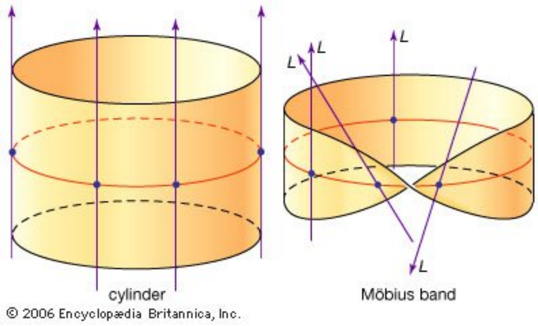
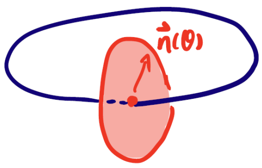
This note reviews some constructions and consequences of the anomaly polynomial for the lattice symmetry and its interplay with other global symmetry. This is relevant to the understanding of deconfined quantum criticality as the anomalous gapless boundary of certain symmetry protected topological state.

INTRODUCTION

The anomaly polynomials are constructed by characteristic classes which measure certain twist of the fields over the base manifold. An intuitive picture for the "twist" is the cylinder and Möbius,



they both have a circle in the middle, but at each point of the circle, the arrow points to different directions, i.e. all arrows point towards up for the cylinder and the arrows rotate by π for the Möbius strip. These two may correspond to the configurations of a non-linear sigma model with both base manifold and target manifold being S^1,



$$\mathbf{n}: S^1 \to S^1.$$

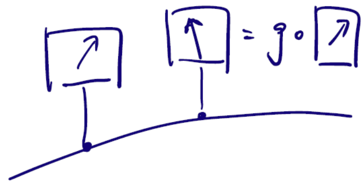
The Möbius strip cannot be continuously deformed to the cylinder, but if the arrow rotates 2π, the configuration can be continuously deformed to the cylinder, this distinction is measured by $H^1(S^1, \mathbb{Z}_2) = \mathbb{Z}_2$. The base circle and the arrow together (with arbitrary twist) form the so-called "bundle", the line bundles of a circle look like the infinite cylinder or the infinite Möbius strip,

$$L_{\text{cyl}}: S^1 \times \mathbb{R} \xrightarrow{\pi} S^1$$
$$L_{\text{Mö}}: ([0, 2\pi] \times \mathbb{R}/\sim) \xrightarrow{\pi} S^1, \quad (0, t) \sim (2\pi, -t),$$

in particular, the cylinder is the trivial line bundle while the Möbius strip is a non-trivial line bundle and the first Stiefel-Whitney class of the line bundles distinguish them, $w_1(L) \in H^1(S^1, \mathbb{Z}_2)$.

In general, the first Stiefel-Whitney class measures whether the bundle is orientable, namely whether all the fibers can be coherently oriented, indeed, the Mö is unorientable while the cylinder is orientable. The second Stiefel-Whitney class tells whether the spin structure can be put on the manifold. For the real vector bundles (fiber is $\mathbb{R}^n$), one needs the Euler class as well as all its Stiefel-Whitney and Pontryagin classes to tell whether the bundle is trivial, but for complex vector bundles (fiber is $\mathbb{C}^n$), one only needs the Chern class.

The transition functions g_{ij} are $\mathsf{GL}(n, \mathbb{R})$ or $\mathsf{GL}(n, \mathbb{C})$ for the real or complex vector bundles. The more physically relevant ones are the principle bundles with transition function in G.



The relation between the gauge connection in gauge theory and the transition function of the G-bundle is,

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} d g_{ij}. \tag{0.1}$$

For the $\mathsf{U}(1)$ gauge theory,

$$g_{ij} = \exp\{(\mathbf{i} f_{ij})\}, \quad A \to A + df \tag{0.2}$$

which gives the familiar form of the $\mathsf{U}(1)$ gauge transformation. One interesting question is that whether we can lift the G-bundle to $\tilde{G}$-bundle with larger symmetry group, the obstructions are measured by various characteristic classes.

Back to physics, we would like to study whether a gauge theory has the 't Hooft anomaly of certain global symmetry. The strategy is attempting to gauge that symmetry, and see whether the term is gauge invariant, if not, it can be cured by a higher dimensional term, this is called anomaly inflow. If the system has that anomaly, the IR phase would be 1) spontaneously symmetry breaking 2) topological order 3) gapless.

Album-bundle and cowability An intuitive understanding of the previous mathematical discussion would be: thinking of the bundle as an album, and the base manifold is the place you visited, the fibers are the photos you took at different positions. You would expect the two photos do not change too much if the positions are close to each other, and there is a transformation acting on all the pixels of one photo to reproduce the other (e.g. clouds move a little bit), that is the transition function. We call it an album-bundle.

The album-bundle has a property: whether we can photoshop a cow in the photo, i.e. the cowability. If you have an album of photos took in the Doyle park, that would be possible to ps a cow in the photos, but UTC-bundle is non-cowable. We may use a $\mathbb{Z}_2$-valued cow-class to measure the cowability.

The cow-class measures the obstruction to lift the transition function from, say clouds motion to the cow motion. The cow-class of the UTC-bundle is non-trivial, thus the lifting is obstructed, while the Doyle bundle is obstruction-free.

ANOMALY OF 1+1D SPIN-$\frac{1}{2}$ CHAIN

The 1+1d spin-$\frac{1}{2}$ chain has mixed anomaly between the $\mathsf{SO}(3)$ spin rotation symmetry and the translation symmetry. This was known by the Lieb-Schultz-Mattis theorem [1], which states that an insulator with half-odd-integer spin per unit cell cannot have a trivial gapped ground state: in 1+1D the ground state must either break the translational symmetry or be gapless, while in higher dimensions the system may also spontaneously break the $\mathsf{SO}(3)_s$ spin rotation symmetry or support topological order.

We would like to review a field theory description presented in Ref. [2], also see Ref. [3]. The gapless phase of the spin-$\frac{1}{2}$ chain can be described by the $\mathbb{C}P^1$ model with θ-term at $\theta = \pi$,

$$\mathcal{L} = |(\partial_\mu - \mathbf{i}a_\mu)z_\alpha|^2 + \mathbf{i}\theta\frac{f}{2\pi}, \quad \theta = \pi, \tag{0.3}$$

where z_α with $\alpha = 1, 2$ is a complex scalar transforming in the projective $S = 1/2$ representation of spin-rotation group $\mathsf{SO}(3)_s$. a_μ is the dynamical gauge field and $f = da = \epsilon_{\mu\nu}\partial_\mu a_\nu$. The Néel order parameter is $z^\dagger\sigma^i z$, and the VBS order parameter is $f = da$. The lattice translation symmetry will act internally on the field contents,

$$T_x : z \to \mathbf{i}\sigma^y z^\star, \quad a \to -a. \tag{0.4}$$

Note that $T^2 z_\alpha = -z_\alpha = \mathsf{U}(1)_\pi z_\alpha$, i.e. π-rotation in the $\mathsf{U}(1)$ gauge group, and thus T_x^2 acts trivially on the physical operators, namely T_x is a $\mathbb{Z}_2$ symmetry.

To show there is a mixed 't Hooft anomaly between the two global symmetries, we are trying to gauge these symmeties. Since the translation symmetry T_x acts internally as $\mathbb{Z}_2^x$, we can safely try to gauge this symmetry as well as the $\mathsf{SO}(3)_s$ spin rotation symmetry. As discussed in the previous section, we are trying to also include the connection for the global symmetries in the transition function, i.e. lift $g_{ij} : U_{ij} \to \mathsf{U}(1)$ to $\tilde{g}_{ij} : U_{ij} \to \mathsf{U}(1) \vee G_1 \vee G_2$, where $\vee$ denotes certain combination of these groups.

Since T_x^2 is the π-rotation of the $\mathsf{U}(1)$ group, when gauging the translation $\mathbb{Z}_2^x$ symmetry, the bosonic field will see the transition function in $\mathsf{Pin}_-(2)$ as reviewed in the appendix.

Together with the spin rotation symmetry, the bosonic field will see the transition function in,

$$G = \frac{\mathsf{Pin}_-(2) \times \mathsf{SU}(2)}{\mathbb{Z}_2} \tag{0.5}$$

The $\mathsf{Pin}_-(2)$ comes from the combination of the $\mathsf{U}(1)$ gauge group and the $\mathbb{Z}_2^x$ lattice translation symmetry which we are trying to gauge, the $\mathsf{SU}(2)/\mathbb{Z}_2 \cong \mathsf{SO}(3)$ is the spin rotation symmetry.

However, when $\mathbb{Z}_2^x$ is gauged the θ term is no longer well-defined since it is odd under $\mathbb{Z}_2^x$, the resolution is to think the theory is on the boundary of a 2+1d SPT bulk. We need to extend the G-bundle into the 3d bulk and the extension should not depend on the bulk we choose.

We first see when the G-bundle satisfies the cocycle condition. The obstruction for lifting $\mathsf{O}(2)$ to the $\mathsf{Pin}_-(2)$ is measured by $w_1^2[g] + w_2[g]$, for lifting $\mathsf{SO}(3)$ to $\mathsf{SU}(2)$ is measured by $w_2[s]$. And the g and s (g denotes the gauge part, s denotes the spin part) combining together is obstruction free, this means,

$$w_1^2[g] + w_2[g] = w_2[s]. \tag{0.6}$$

The Ref. [2] proposes the anomaly is captured by

$$\mathbf{i}\pi \int_3 w_1^3[g] + w_1[g]w_2[s] \tag{0.7}$$

By the obstruction free condition, we have,

$$\begin{aligned}
&w_1^3[g] + w_1[g]w_2[s] \bmod 2\\
=&w_1[g](w_2[g] - w_2[s]) + w_1[g]w_2[s] \bmod 2\\
=&w_1[g]w_2[g] \bmod 2\\
=&dw_2[g]/2 \bmod 2
\end{aligned}$$

And when evaluating on a closed 3-manifold, the anomaly vanishes, meaning that the extension is independent of the bulk we chose.

We denote $x \equiv w_1[g] \in H^1(M^2, \mathbb{Z}_2)$ as an element in first Stiefel Whitney class, this is also the $\mathbb{Z}_2^x$-gauge field. The first term in Eq. 0.7 is the $\mathbb{Z}_2^x$ anomaly and the second term is the mixed anomaly between $\mathbb{Z}_2^x$ and $\mathsf{SO}(3)_s$ spin rotation symmetry.

[*This part is probably not rigorous, I'm using a different approch from the Ref. [2]*] It remains to show that the anomaly term reduces to the θ term when not gauging the $\mathbb{Z}_2^x$ and $\mathsf{SO}(3)_s$. The second term in Eq. 0.7 vanishes automatically, the first term can be replaced with

$$w_1^3[g] = w_1[g]w_2[g] \bmod 2 = w_1[g]c_1[g] \bmod 2, \quad (0.8)$$

and imagine thread a flux for x in the virtual dimension, this results in,

$$\mathbf{i}\pi\left(\int_1 x\right)\frac{f}{2\pi} = \mathbf{i}\pi\frac{f}{2\pi}. \quad (0.9)$$

reproduces the θ term.

Acknowledgements My warmest thanks to Prof. John McGreevy for teaching us the algebraic topology and many applications in physics.

APPENDIX

$\mathsf{Pin}_+(2)$ and $\mathsf{Pin}_-(2)$ [4]

We first introduce the basis, $\{e_i\}$, they satisfy,

$$\{e_i, e_j\} = 2\eta^{ij}\mathbb{1}, \quad \eta^{ij} = \mathbb{1}_r \oplus (-\mathbb{1}_s). \quad (0.10)$$

Then we define $\mathsf{Pin}_\pm(d)$ according to whether $e_i^2 = +1$ or $e_i^2 = -1$.

$\mathsf{Pin}_+(2)$ We define

$$\mathcal{O}(\alpha) = \cos(\alpha)e_1 + \sin(\alpha)e_2, \quad (0.11)$$
$$\mathcal{E}(\alpha) = \cos(\alpha) + \sin(\alpha)e_1e_2 \quad (0.12)$$

where $e_i^2 = +1$, then $\mathsf{Pin}_+(2) = \{\mathcal{O}(\alpha)\} \cup \{\mathcal{E}(\alpha)\}$, each component is isomorphic to the circle. The group multiplication is,

$$\mathcal{E}(\alpha)\mathcal{E}(\beta) = \mathcal{E}(\alpha+\beta) \quad \mathcal{O}(\alpha)\mathcal{E}(\beta) = \mathcal{O}(\alpha+\beta)$$
$$\mathcal{E}(\alpha)\mathcal{O}(\beta) = \mathcal{O}(\alpha-\beta) \quad \mathcal{O}(\alpha)\mathcal{O}(\beta) = \mathcal{E}(-\alpha+\beta)$$

and $\mathcal{O}(\alpha)\mathcal{O}(\alpha) = +1$.

$\mathsf{Pin}_-(2)$ Similarly, we define

$$\mathcal{O}(\alpha) = \cos(\alpha)e_1 + \sin(\alpha)e_2, \quad (0.13)$$
$$\mathcal{E}(\alpha) = \cos(\alpha) + \sin(\alpha)e_1e_2 \quad (0.14)$$

where $e_i^2 = -1$, then $\mathsf{Pin}_-(2) = \{\mathcal{O}(\alpha)\} \cup \{\mathcal{E}(\alpha)\}$, each component is isomorphic to the circle. The group multiplication is,

$$\mathcal{E}(\alpha)\mathcal{E}(\beta) = \mathcal{E}(\alpha+\beta) \quad \mathcal{O}(\alpha)\mathcal{E}(\beta) = \mathcal{O}(\alpha-\beta)$$
$$\mathcal{E}(\alpha)\mathcal{O}(\beta) = \mathcal{O}(\alpha+\beta) \quad \mathcal{O}(\alpha)\mathcal{O}(\beta) = \mathcal{E}(\alpha-\beta+\pi)$$

and $\mathcal{O}(\alpha)\mathcal{O}(\alpha) = \mathcal{E}(\pi)$.

Obstruction

Let $\tilde{G}$ be a simple simply-connected group. Fix a closed spin four-manifold X. Consider a G-bundle on it with $G = \tilde{G}/\mathbb{Z}_k$. The obstruction to lift the G-bundle to $\tilde{G}$-bundle is measured by,

$$w_2 \in H^2(X, \mathbb{Z}_k) \quad (0.15)$$

The $\mathbb{Z}_k$ is the center of the group,

G	A_n	B_n	C_n	D_{2n}	D_{2n+1}	E_6	E_7
Z	$\mathbb{Z}_{n+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_3$	$\mathbb{Z}_2$

also [5],

$$\mathsf{Spin}(n),\ n = 2d+1, \quad \mathbb{Z}_2$$
$$\mathsf{Spin}(n),\ n = 4d+2, \quad \mathbb{Z}_4.$$

Given an unoriented d dimensional manifold, the obstruction for lifting to $\mathsf{Pin}_\pm(d)$ is measured by [6]

$$w_2(X), \quad \text{for } \mathsf{Pin}_+(d),$$
$$w_2(X) + w_1^2(X), \quad \text{for } \mathsf{Pin}_-(d),$$

If X is orientable, then $w_1(X) = 0$, the two conditions coincide and reduce to the condition that X admit a Spin structure (w_2).

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