Gapped domain walls, symmetry-protected topological phases, and fault-tolerant logical gates in topological color code

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We discuss a connection between the gapped domain walls, symmetry-protected topological phases, and fault-tolerant logical gates in topological color codes based on [Phys. Rev. B 91, 245131]. It was found that applying a d-dim transversal operator leads to $d-1$-dim excitations characterized by bosonic symmetry-protected topological (SPT) wave functions, and these SPT-excitations can be realized as transparent gapped domain walls in the color code. The connection may be generalized to a large class of topological quantum codes and topological quantum field theories.

I. INTRODUCTION

Symmetry-protected topological (SPT) quantum phases are described by short-range entangled states that cannot be smoothly connected to trivial product states in the presence of symmetries[1–3]. The study of SPT phases has enriched quantum many-body physics in many aspects, yet their implication in quantum information science is less explored. In Ref.[4, 5], Yoshida explores an intriguing connection between SPT phases and fault-tolerant logical gates in topological quantum codes. Such a connection is notable since one may apply the known classification of SPT phases to classify or construct fault-tolerant logical gates, which is essential in topological quantum computation. Furthermore, Yoshida also pointed out that gapped boundaries and domain walls[6], another important subject in the study of topological order, can be constructed given the knowledge of fault-tolerant logical gates. As such, a connection between SPT phases, fault-tolerant logical gates, and gapped boundaries can be established.

In this report, we will provide a short review regarding the aforementioned connection. We will mainly follow Ref.[4], which focuses on d-dim topological color codes[7, 8], and the generalization to the quantum double model discussed in Ref.[5] will not be discussed here.

II. TOPOLOGICAL COLOR CODE

A two-dimensional color code can be defined on any three-valent and three-colorable lattice, where each vertex accommodates a qubit. One common choice is the hexagonal lattice with the Hamiltonian defined as

$$H = -\sum_P S_P^{(X)} - \sum_P S_P^{(Z)}. \tag{1}$$

P labels a plaquette, and $S_P^{(X)}$, $S_P^{(Z)}$ are products of Pauli-X, Z operators acting on all qubits on the plaquette P (see Fig.1). This is a stabilizer Hamiltonian since every term commutes with each other, and correspondingly, a ground state $|\psi\rangle$ satisfies $S_P^{(X)}|\psi\rangle = S_P^{(Z)}|\psi\rangle = |\psi\rangle$. Being a topological code, H supports anyonic excitations, which live on two ends of string operators. To construct these excitations, we now assign colors A, B, and C such that two plaquettes that share an edge have different colors. It follows that an edge can also have the

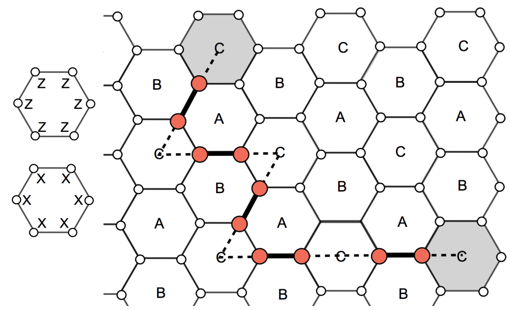

FIG. 1: Left: local terms $S_P^{(Z)}$ and $S_P^{(X)}$ in the color code Hamiltonian (Eq.1). Right: a string operator γ^{AB} creates a pair of excitations on its two ends (the shaded plaquettes).

color AB, BC, or CA that is determined by its two neighboring plaquettes. For a set of edges of color AB which form a one-dimensional line γ^{AB} (see Fig.1), one can define

$$\overline{X^{AB}}|_{\gamma^{AB}} = \prod_{j\in\gamma^{AB}} X_j, \quad \overline{Z^{AB}}|_{\gamma^{AB}} = \prod_{j\in\gamma^{AB}} Z_j, \tag{2}$$

and it is not hard to see that such string operators commute with all the terms in H, except for the two plaquette terms on their boundary. Correspondingly, applying $\overline{X^{AB}}|_{\gamma^{AB}}$ and $\overline{Z^{AB}}|_{\gamma^{AB}}$ on a ground state $|\psi\rangle$ creates the magnetic fluxes m_C and electric charges e_C respectively, which can be expressed as

$$\overline{X^{AB}}|_{\gamma^{AB}} \to m_C, \quad \overline{Z^{AB}}|_{\gamma^{AB}} \to e_C. \tag{3}$$

Similarly, one can construct other types of string operators that create anyons: $\overline{X^{BC}}|_{\gamma^{BC}} \to m_A$, $\overline{Z^{BC}}|_{\gamma^{BC}} \to e_A$, $\overline{X^{CA}}|_{\gamma^{CA}} \to m_B$, and $\overline{Z^{CA}}|_{\gamma^{CA}} \to e_B$. Importantly, the excitations are not independent from each other since applying a single Pauli-X/Z creates the composite excitations $m_A m_B m_C / e_A e_B e_C$, implying the existence of the fusion channel $m_A \times m_B \times m_C = 1$ and $e_A \times e_B \times e_C = 1$. In addition to the string operators, there also exist membrane-like operators as follows. Considering the Hadamard operator $\mathcal{H}$, which exchanges a Pauli-X and a Pauli-Z, i.e. $\mathcal{H}X\mathcal{H}^\dagger = Z$ and $\mathcal{H}Z\mathcal{H}^\dagger = X$, one can define the membrane operator

$$\overline{\mathcal{H}} = \prod_j \mathcal{H}_j, \tag{4}$$

which commutes with H but has the non-trivial operation for exchanging magnetic fluxes and electric charges. To construct another non-trivial membrane operator, one first divides the (bipartite) hexagonal lattice into two sublattices T and T^c so that the nearest neighbors of every site in T belong to T^c and vice versa, and then a membrane operator $\overline{R}$ can be defined as

$$\overline{R} = \prod_{j\in T} R_j \prod_{i\in T^c} \left(R_j\right)^{-1}, \tag{5}$$

where $R = \sqrt{Z} = \mathrm{diag}(1, i)$ is a phase gate which exchanges Pauli-X and Y as $RXR^\dagger = Y$ and $RYR^\dagger = -X$. Crucially, although $\overline{R}$ does not commute with the Hamiltonian H, it commutes with the projector to the ground state subspace of H, and can implement the nontrivial operation on anyons:

$$e_A \to e_A, e_B \to e_B, m_A \to m_A e_A, m_B \to m_B e_B. \tag{6}$$

Having introduced the basics of the color code Hamiltonian, we now discuss how SPT orders naturally arise in the application of membrane operators.

III. SPT ORDER FROM FAULT-TOLERANT LOGICAL GATES

Here we will show that restricting a membrane operator in a subregion V induces a loop-like excitation (on its boundary ∂V) characterized by a non-trivial SPT order. First, let's consider the flux-free subspace $\mathcal{H}_{no-flux}$ where $|\psi\rangle \in \mathcal{H}_{no-flux}$ satisfies $S_P^{(Z)}|\psi\rangle = |\psi\rangle$. One can define an excitation basis to encode the location of excitations in $S_P^{(X)}$:

$$S_{P_j}^{(X)}\left|\tilde{p}_1,\cdots,\tilde{p}_{n_0}\right\rangle = (1-2p_j)\left|\tilde{p}_1,\cdots,\tilde{p}_{n_0}\right\rangle, \tag{7}$$

where $\tilde{p}_j$ can be 0 or 1, corresponding to the absence or presence of the excitation on the plaquette $\tilde{p}_j$, and n_0 denotes the total number of plaquettes on the lattice. Since a ground state $\left|\psi_{gs}\right\rangle$ is invariant under the application of the membrane operators $\overline{R}$, restricting $\overline{R}$ on a subregion V creates a loop-like excitation on the boundary of V (see Fig.2), and the corresponding wave function $|\psi_V\rangle = \overline{R}|_V\left|\psi_{gs}\right\rangle$ can be conveniently expressed in the excitation basis as

$$|\psi_V\rangle \to |\psi_{\partial V}\rangle \otimes \left|\tilde{0},\cdots,\tilde{0}\right\rangle, \tag{8}$$

where $|\psi_{\partial V}\rangle$ denotes the boundary excitations on the plaquettes $A_1, B_1, \cdots A_n, B_n$:

$$|\psi_{\partial V}\rangle = \sum_{\tilde{p}_{A_1},\tilde{p}_{B_1},\cdots\tilde{p}_{A_n},\tilde{p}_{B_n}} \lambda(\{\tilde{p}_j\})\left|\tilde{p}_{A_1},\tilde{p}_{B_1},\cdots\tilde{p}_{A_n},\tilde{p}_{B_n}\right\rangle \tag{9}$$

and $\left|\tilde{0},\cdots,\tilde{0}\right\rangle$ denotes the rest (un-excited) plaquettes. Notably, the state $|\psi_{\partial V}\rangle$ is the exact ground state wave function of a 1d cluster state Hamiltonian, which exhibits a non-trivial $Z_2\times Z_2$ SPT order. While it is not hard to see that the $Z_2\times Z_2$ symmetry arises simply from the number parity conservation

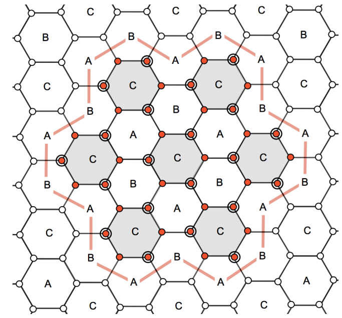

FIG. 2: A loop-like excitation with the $Z_2\times Z_2$ SPT order (Eq.9) supported on the boundary of a membrane operator. The phase gate operators $R(R^{-1})$ are applied on filled circles (filled double circles).

of the electric charges e_A and e_B: $\mathcal{N}_A = \mathcal{N}_B = 0 \bmod 2$, it remains non-trivial that the boundary excitation corresponds to an SPT phase. The central idea is that the preparation of the boundary loop excitation cannot be obtained by just applying a local unitary transformation localized on the boundary. To see this, one can imagine transporting a magnetic flux m from the region outside of V to inside of V. Based on Eq.6, the m flux will be transformed into a pair of e charge and m flux, and this implies one must apply a unitary operation on all qubits in V to realize the boundary excitation, hence giving rise to the non-trivial SPT order. In particular, the application of the R phase gate is essentially a symmetry-protected quantum circuit for preparing the SPT wave functions.

IV. GAPPED DOMAIN WALLS FROM FAULT-TOLERANT LOGICAL GATES

We have seen that applying a membrane operator restricted in a subregion V for a ground state creates fluctuating charges with a non-trivial SPT order on the boundary of V. Now we show that applying such operators to transform Hamiltonian allows to create gapped domain walls, which establishes a connection between the classification of gapped domain walls and fault-tolerant logical gates.

To start, one can split the lattice into the left part and the right part, and perform a transformation on the color code Hamiltonian using the Hadamard gates restricted in the right of the lattice (i.e. $\overline{\mathcal{H}}|_R = \prod_{j\in R}\mathcal{H}_j$). It follows that the transformed Hamiltonian $\tilde{H} = \overline{\mathcal{H}}|_R H(\overline{\mathcal{H}}|_R)^\dagger$ reads $\tilde{H} = H_L + H_R + H_{LR}$, where $\tilde{H}$ differ from H only in the terms localized on the boundary between L and R. Crucially, since $\overline{\mathcal{H}}|_R$ implements a unitary transformation, $\tilde{H}$ remains gapped, and H_{LR} can be regarded as a gapped domain wall, across which anyons exchange m-fluxes and e-charges: $(e_A|m_A), (m_A|e_A), (e_B|m_B), (m_B|e_B)$. Note that such a domain wall is transparent in the sense that a single anyon cannot condense on the domain wall. One can follow a similar strategy to construct another gapped domain wall using the phase gate R, which induces the following anyon exchange rule $(m_A|e_Am_A), (e_A|e_A), (m_B|e_Bm_B), (e_B, m_B)$ (see Fig.3). More broadly, one can show that the membrane opera-

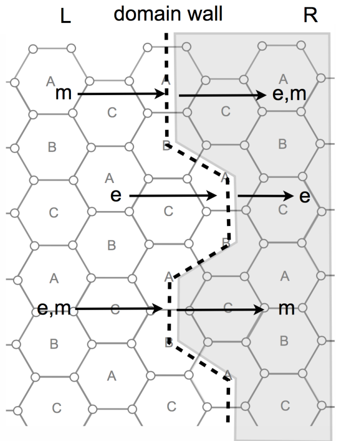

FIG. 3: A transparent gapped domain wall obtained by restricting the membrane operator $\overline{R}$ in the right.

tors associated with non-trivial automorphisms among anyon labels always lead to transparent gapped domain walls in (2 + 1)-dimensional TQFTs.

V. SUMMARY AND DISCUSSION

We have presented a short review regarding the connection between SPT phases, gapped domain walls, fault-tolerant logical gates in the two-dimensional topological color code. Such an observation in fact can be generalized to the d-dim color code[4]. In particular, applying R_d phase gates ($R_d = \mathrm{diag}(1, \exp(i\pi/2^{d-1}))$) on all qubits in a connected subregion V creates $d-1$ dim excitations characterized by a bosonic SPT order with $(Z_2)^{\otimes d}$ symmetry, and this SPT order in turn characterizes the gapped domain walls in the d-dim color codes.

Finally, we note that the aforementioned connection has been generalized to d-dim quantum double models[5], where it was found that using d-cocycle functions, one can construct the gapped boundaries/domain walls, fault-tolerantly implementable logical gates, and excitations characterized by an SPT order.

Yoshida's works[4, 5] have motivated some interesting questions. A natural future direction is generalizing the discussion of global on-site symmetries of SPT orders to q-form symmetries, where symmetry operators are codimension-1 objects, and exploring the implication to gapped boundaries and fault-tolerant logical gates. Similarly, one may employ SPT orders with fractal-like symmetries to explore novel types of gapped boundaries and logical gates.

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