How to (in principle) measure topological data?

Shubham Parashar[1]
[1]Department of Physics, University of California at San Diego, La Jolla, CA 92093

INTRODUCTION

The topological order of a 3D system is fully characterized by a unitary modular tensor category (UMTC) which specifies the fusion and braiding properties of the system. The basic data defining a UMTC are the R-symbols and F-symbols. The modular data defining a UMTC are the S and T matrices. The S-matrix is the invariant associated with a single figure-8 stand and T-matrix is a invariant if the strands of the figure-8 are colored by two objects. The modular data contains the rank, fusion rules (from Verlinde), dimensions of the irreps, central charge etc. Unfortunately the modular data is incomplete.[1] This (very) short paper mentions some experiments to probe UMTC.[2]

EXPERIMENTS

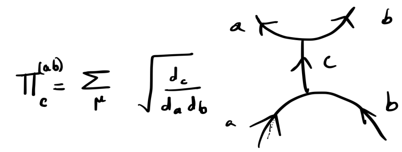
As an (in principal) experimenter we shall assume that we can (in principal) do certain things to anyons. Some of those things are: 1) we can localize the anyon - this is done by introducing a pinning potential at some site, measure the charge of the anyon, move around anyons in the bulk - this can be done by (non-local) quantum teleportation or moving the position of the pinning potential and split anyons - this is done by adiabatically changing the pinning potential or interferometry.

Fusion rules and Quantum Dimensions

The simplest thing to do is to just measure the collective charge of a pair of anyons, that is given the vacuum state 0, we first create a pair of $a, \bar{a}$ and $b, \bar{b}$ move the pair apart and then measure the outcome of the experiment to find the probability that a and b fuse to c. This is given by

$$p_{ab}(c) = \left| \Pi_c^{(ab)} \middle| \bar{a}, a; 0 \right\rangle |b, \bar{b}; 0\rangle \Big|^2$$
$$= \left\langle \bar{a}, a; 0 \middle| \left\langle b, \bar{b}; 0 \middle| \Pi_c^{(ab)} \middle| \bar{a}, a; 0 \right\rangle \middle| b, \bar{b}; 0 \right\rangle = N_{ab}^c \frac{d_c}{d_a d_b}$$

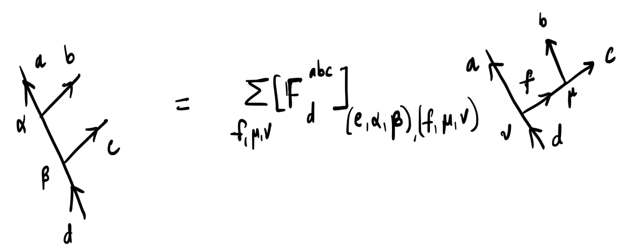
where $\Pi_c^{(ab)}$ is defined by,



and d_a, d_b, d_c are the quantum dimensions of charges a, b, c. Repeating this experiment many times for all possible values of a, b will help us infer the fusion coefficients and quantum dimensions.

Associativity

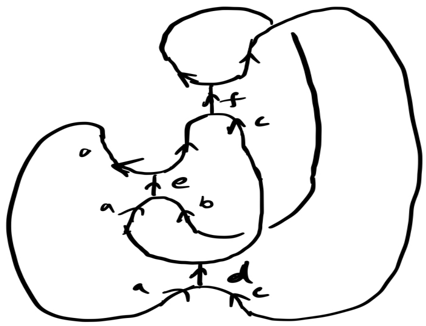
In this experiment we calculate the F-symbols which are defined by,



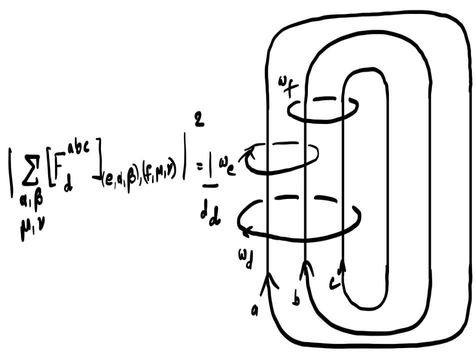
where the greek symbols are the dimensions of the fusion spaces. The experiment is done in two steps, first we we do an initial setup as follows: create charges $d, \bar{d}$ and pull them apart, now split d into charges e, c and pull them apart, and finally split and separate e into a, b. Now, the second setup involves doing a series of measurements: measure the charge of $b - c$, measure the charge charge of $a - b$ and now go back to the first step. The probability that $a - b$ has charge f given $a - b$ had charge e is

$$p_{a(bc);d}(f \mid e) = \left| \Pi_f^{(bc)} \middle| a, b; e \right\rangle |e, c; d\rangle |d, \bar{d}; 0\rangle \Big|^2$$
$$= \left| \left[F_d^{abc} \right]_{ef} \right|^2$$

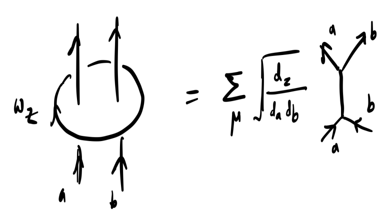
The first step can be diagramatically represented as



which is equivalent to



where the ω loops are defined as



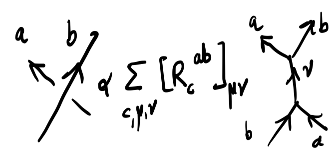
Repeating this experiment many times for different values of a, b, c and d will give magnitudes of all F-symbols. The phase information may be found via consistency relations like pentagon equation.

Braiding

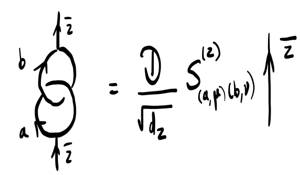
The braiding experiment is carried out by create-separating charges $(a, \bar{a})$ $(b, \bar{b})$, braiding them around each other and measuring the charge of $\bar{a} - a$. The probability that the pair $\bar{a} - a$ has a charge z is given by,

$$p^{(2)}_{ab}(z) = \left| \Pi^{(\bar{a}a)}_{z} R^{ab} R^{ba} \middle| \bar{a}, a; 0 \right\rangle |b, \bar{b}; 0\rangle \Big|^2 = \frac{\mathcal{D}^2 d_z}{d_a^2 d_b^2} \left| S^{(z)}_{\bar{a}b} \right|^2$$

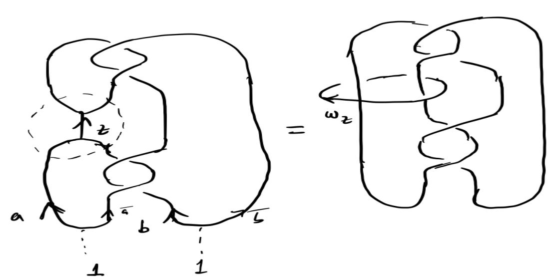
where the R-symbol is defined by



and $S^{(z)}_{ab}$ is the punctured torus S-matrix



The above formulae can be checked using the diagram below,



Last experiment

In 3D a TQFT can be defined in terms of the corresponding UMTC. On a nontrivial surface the TQFT tells us something about the invariants associated with topological operations on the state space given by the mapping class group which is defined as the isotopy equivalence classes of orientation preserving diffeomorphisms of the surface. On a torus, a canonical ground state can be defined in terms of charge a (flux threading the cycle) and an ordered pair (l, m) of generating cycles on the surface, denoted by $|\Phi_0\rangle_{(l,m)}$. The mapping class group of the torus $\cong \mathrm{SL}(2, \mathbb{Z})$, relates any two choices (l, m) and (l', m') of generating pairs of cycles of the torus as

$$(l, m) = \mathfrak{q}\,(l', m') = (\alpha l' + \beta m', \gamma l' + \delta m')$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ and $\alpha\delta - \beta\gamma = 1$.

$$q \cong \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

is be generated by the two elements

$$\mathfrak{s} \cong \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathfrak{t} \cong \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

which satisfy group relations $(5\mathfrak{t})^2 = \mathfrak{s}^2$ and $\mathfrak{s}^4 = 1$. For a general modular transformation $\mathfrak{q}$ the projective representation of its action is given by

$$|\Phi_a\rangle_{(l,m)} = \sum_b Q_{ab}\, |\Phi_b\rangle_{(l',m')}$$

where Q can be expressed in terms of the S and T matrices in the same manner that q is generated from $\mathfrak{s}$. The experiment to measure Q is carried out by first measuring the charge around the cycle m and m'. The probability that the measurement around the cycle m' will have outcome b, given that measurement around the cycle m had value a is

$$p_{\mathrm{q}}(b \mid a) = |Q_{ab}|^2$$

Question: Is it possible to distinguish the categories in Mignard and Schaenberg paper by considering mapping class groups on a punctured torus?

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[1] It was thought that the modular data is all the invariants of a modular tensor category until 2017 when Mignard and Schauenberg found that modular categories $Z(Vec_G^{\omega})$ for $G = Z/q \rtimes Z/p$, where p, q are primes and $p|q-1$ cannot be explicitly determined from the modular data. Bonderson et. al. showed that the $W-$matrix related to the whitehead link (and to the punctured S-matrix) and T-matrices distinguished the modular tensor categories in the counter example.

[2] Parsa Bonderson, "Measuring Topological Order," arxiv:2102.05677 1

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