Notes on TQFT from M-theory

In this short note, we briefly describe how to acquire TQFT from compactifying $M5$ branes over 3-manifold $\mathcal{M}$ following [1].

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1. Overview

In this short note, we want to explain briefly how one can acquire TQFT from compactifying $M5$ branes over 3-dimensional manifolds $\mathcal{M}$ in M-theory.

The setup is canonical, we choose the 11-dim space time to by $\mathbb{R}^{1,2} \times (T^*\mathcal{M}) \times \mathbb{R}^2$ and place the two coincident $M5$ branes on $\mathbb{R}^{1,2} \times \mathcal{M}$. And we will denote the resulting 3d theory as $T[\mathcal{M}]$.

We will divide the sections into two parts. First, notice that compactifying higher dimensional theories in general does not give theories which are IR gapped, let alone TQFT. Therefore, it is important to determine which compact manifolds $\mathcal{M}$ would lead to TQFT. We will explain this in the first part. Second, once we know we get a TQFT after compactification, we wish to figure out a way to extract data from the geometry of $\mathcal{M}$. We will discuss this in the second part.

The discussions require some background on various stuffs, which we will introduce as we run into it. However, we will omit many subtleties[1] and readers who are interested can read the original article [1] or discuss me with email. Also, I am not using the provided tex template as I found many people did not use it in the last quarter. The author also thanks John McGreevy for pointing out this paper to me half a year ago.

2. When do we expect TQFT?

To answer this question when we can get TQFT, we start our story with an important observable in the supersymmetric theory, the supersymmetric index, counting only the protected operators. Those indices are invariant under the renormalization group flow, therefore are useful tools to probe the IR theory. Usually, those states counted by the index are counted by supersymmetry or by topology. In 3d, the super conformal index $\mathcal{I}_{\text{sci}}$ is defined as

$$\mathcal{I}_{\text{sci}}(x) = \text{Tr}_{\mathcal{H}(S^2)}(-1)^R x^{\frac{R}{2}+j_3}$$

where $R \in \mathbb{Z}$ is the charge under UV $U(1)_R$ symmetry, $j_3 \in \mathbb{Z}/2$ is the charge of $SO(3)$ isometry of S^2, $x = e^{-\beta}$ where β is the perimeter of Euclidean time circle S^1. This index counts the BPS local operators preserving 2 supercharges.

There is another index, the topologically twisted refined index $\mathcal{I}_{\text{top}}$, counting the supersymmetric ground states on a topologically twisted S^2 with unit background magnetic flux of the $U(1)_R$ symmetry:

$$\mathcal{I}_{\text{top}}(x) = \text{Tr}_{\mathcal{H}(S^2)}(-1)^R x^{j_3}.$$

The above two indices can be expressed in terms of Bethe vacua. To give a brief review of the Bethe vacua [2], let's start with 2d gauge theories with $\mathcal{N} = (2,2)$ supersymmetry. We focus on the Coulomb branch of the theory, and deform the theories by adding twisted mass m, which amounts to couple the matter fields to a background vector multiplet with only the bottom scalar being non-zero and constant. Notice that we can use gauge symmetry to rotate m into the Cartan

[1] For instance, we will not discuss the condition such that $T[\mathcal{M}]$ is a unitary TQFT.

of the gauge group. For generic m, all matter fields would become massive, thus can be integrated out. The resulting contribution is an F-term and is 1-loop exact. On the Coulomb branch, the massive modes in the gauge multiplet would also contribute. Combine the two results we get the effective twisted superpotential $\tilde{\mathcal{W}}(\sigma)$. The Bethe vacua are defined as the local minimum α of the twisted superpotential [2]

$$\exp\left(\frac{\partial\tilde{\mathcal{W}}}{\partial\sigma^i}\right) = 1.$$

In 3d, if we compactify the theory over S^1, each multiplet leads to an infinite tower of KK modes. We can integrate over the contributions from these massive KK modes; this leads to definition of effective twisted superpotential and the Bethe vacua for 3d theory.

Then, the two indices can be expressed as

$$\mathcal{I}_{\text{sci}}(x) = \sum_{\alpha:\text{Bethe-vacua}} \mathbb{B}^\alpha(x)\mathbb{B}^\alpha(x^{-1})^*,$$
$$\mathcal{I}_{\text{top}}(x) = \sum_{\alpha:\text{Bethe-vacua}} \mathbb{B}^\alpha(x)\mathbb{B}^\alpha(x^{-1}),$$

where $\mathbb{B}(x)$ is the holomorphic block, computing the partition function on $\mathbb{R}^2 \times S^1$ with an asymptotic boundary condition determined by the choice of a Bethe-vacuum α.

The key observation in [1] is that $\mathcal{I}_{\text{sci}}(x) = \mathcal{I}_{\text{top}}(x)$ hints the theory would flows to a topological theory in the IR. This is quite natural, since $\mathcal{I}_{\text{sci}}(x) = \mathcal{I}_{\text{top}}(x)$ suggests there are no SUSY protected states except the ones are also protected by topology.

$\mathcal{I}_{\text{sci}}(x) = \mathcal{I}_{\text{top}}(x)$ translates into $\mathbb{B}^\alpha(x) = \mathbb{B}^\alpha(x)^*$ for all α; and the fact that the theory is acquired from compactifying M5 branes over 3-manifold $\mathcal{M}$ allows us to further relates the above condition to the geometric property of $\mathcal{M}$. For this, we will use the 3d-3d correspondence between 3d theory $T(\mathcal{M})$ and 3d $SL(2,\mathbb{C})$ CS theory on $\mathcal{M}^3$. In particular, there is a one-to-one correspondence between Bethe vacua of $T(\mathcal{M})$ and the irreducible flat connection $\mathcal{A}_\alpha$. By irreducible, we mean their holonomy matrices are not all mutually commuting.

It is claimed in [1] if $\mathcal{M}$ satisfies

- i) there are (non-empty) finitely many irreducible $SL(2,\mathbb{C})$ flat connections on $\mathcal{M}$;
- ii) all of them are gauge equivalent to either $SU(2)$ or $SL(2,\mathbb{R})$ flat connection;

then $\mathbb{B}^\alpha(x) = \mathbb{B}^\alpha(x)^* \implies \mathcal{I}_{\text{sci}}(x) = \mathcal{I}_{\text{top}}(x)$.

To see this, we must use another relation in the 3d-3d dictionary relates the partition function of $SL(2,\mathbb{C})$ and the holomorphic blocks of $T(\mathcal{M})$, the asymptotic expansion $Z^\alpha_{CS\,pert.}$ as $\hbar \to 0$ is equal to the perturbative expansion of homolorphic block $\mathbb{B}^\alpha(q)$ over $S^2 \times_q S^1$ (the S^2 is fibered over S^1 with holonomy $\log q$) in the limit $q \to 1$:

$$Z^\alpha_{CS\,pert.}(\hbar) \simeq \mathbb{B}^\alpha(q := e^\hbar).$$

[2] We do not write $\frac{\partial\tilde{\mathcal{W}}}{\partial\sigma^i} = 0$, because $\tilde{\mathcal{W}}$ are defined up to unphysical shift $\tilde{\mathcal{W}} \to \tilde{\mathcal{W}} - 2\pi i \sum_{i=1}^{\text{rank}G} n_i\sigma^i, n_i \in \mathbb{Z}$.

[3] If we have N M5 branes instead, the 3d theory would be $SL(N,\mathbb{C})$ CS theory.

The expansion $Z^{\alpha}_{CS\,pert.}$ is given by

$$Z^{\alpha}_{CS\,pert.} := \int \frac{D\delta\mathcal{A}}{(gauge)} e^{-\frac{1}{2\hbar}CS[\mathcal{A}^{\alpha}+\delta\mathcal{A};\mathcal{M}]} \to \exp\left(\frac{1}{\hbar}S_0^{\alpha} + S_1^{\alpha} + \cdots + \hbar^n S_{n+1}^{\alpha}\cdots\right) \tag{2.1}$$

where for instance

$$S_0^{\alpha} = -\frac{1}{2}CS[\mathcal{A}^{\alpha},\mathcal{M}]$$

For a $SL(2,\mathbb{R})$ or $SU(2)$ flat connection $\mathcal{A}_{\alpha}$, its complex conjugation $(\mathcal{A}_{\alpha})^*$ satisfies

$$(\mathcal{A}_{\alpha})^* = \mathcal{A}_{\alpha}, \quad for\, SL(2,\mathbb{R}),$$
$$(i\mathcal{A}_{\alpha})^* = (i\mathcal{A}_{\alpha})^T, \quad for\, SU(2).$$

This implies the two flat connection $\mathcal{A}_{\alpha}$ and $(\mathcal{A}_{\alpha})^*$ have the same perturbative expansion in (2.1) as $S_n^{\alpha} = S_n^{\overline{\alpha}}$, which further implies $\mathbb{B}^{\alpha}(x) = \mathbb{B}^{\alpha}(x)^*$.

3. Modular data from the geometry of $\mathcal{M}$

Recently, there has been a huge progress on the calculation of the supersymmetric partition function over various supersymmetric background via localization. In fact, it is even possible to find some organization principles for supersymmetric partition function. For instance, the twisted partition functions over a class of manifold $\mathcal{M}_{g,p}$ (the degree-p S^1-bundles over the genus g Riemann surface Σ_g) can be constructed as [3]

$$Z_{g,p} = \sum_{\alpha} Z^{\alpha}_{g,p} = \sum_{\alpha} (\mathcal{H}^{\alpha})^{g-1}(\mathcal{F}^{\alpha})^p \tag{3.1}$$

where $\mathcal{H}$ is called the handle-gluing operator (as multiplying it increase the genus by 1) and $\mathcal{F}$ is called the fibering operator(as multiplying it increase the deg of fiberation by 1).

The 3d-3d correspondence tells us

$$\mathcal{H}^{\alpha} = \exp(-2S_1^{\alpha}) = 2\mathrm{Tor}[\mathcal{A}_{\alpha}],$$
$$\mathcal{F}^{\alpha} = \exp(iS_0^{\alpha}/(2\pi)) = \exp(-2\pi i CS[\mathcal{A}^{\alpha},\mathcal{M}])$$

where $\mathrm{Tor}(\mathcal{A}_{\alpha})$ is the adjoint Reidemeister torsion of the irreducible $SL(2,\mathbb{C})$ flat connection $\mathcal{A}_{\alpha}$.

If we set $p = 0$, then the partition function over $\mathcal{M}_{g,p}$ reduces to the partition function over $S^1 \times \Sigma_g$, thus compute the ground states degeneracy of $T(\mathcal{M})$ over the Riemann surface Σ_g:

$$\mathrm{GSD}_g = \sum_{\alpha} (2\mathrm{Tor}[\mathcal{A}_{\alpha}])^{g-1}.$$

This relation also hints the anyons are labelled by the irreducible flat connection $\mathcal{A}_{\alpha}$ (and equivalently, the Bethe vacua α) and should has the quantum dimension $2\mathrm{Tor}[\mathcal{A}_{\alpha}]$.

We can also extract the modular $T_{\alpha\beta}$ from the $\mathcal{M}_{g,p}$ partition function by comparing (3.1) with the generic result from TQFT [4]:

$$Z[\mathcal{M}_{g,p}] = \sum_{\alpha} S_{0\alpha}^{2-2g} T_{\alpha\alpha}^{-p},$$

we find

$$(S_{0\alpha})^2 = \frac{1}{2\mathrm{Tor}[\mathcal{A}_\alpha]}, \quad T^\alpha_\alpha = \exp(-2\pi i CS[\mathcal{A}_\alpha]).$$

To extract the full $S_{\alpha\beta}$, we consider quantize the theory $T[\mathcal{M}]$ on 2-torus with two linear independent cycle A and B. We will denote the anyon α wrapping around A/B-cycle as $\mathcal{O}^A_\alpha/\mathcal{O}^B_\alpha$. Those two sets of operators are related by

$$\mathcal{O}^A_\alpha = S^{-1}\mathcal{O}^B_\alpha S.$$

We choose the basis of ground states as

$$|\alpha\rangle = \mathcal{O}^B_\alpha|0\rangle.$$

Then we find

$$\mathcal{O}^B_\alpha|\beta\rangle = \mathcal{O}^B_\alpha\mathcal{O}^B_\beta|0\rangle = \sum_\gamma N^\gamma_{\alpha\beta}|\gamma\rangle,$$
$$\mathcal{O}^A_\alpha|\beta\rangle = (S^{-1}\mathcal{O}^B_\alpha S)|\beta\rangle = \frac{S_{\alpha\beta}}{S_{0\beta}}|\beta\rangle \equiv \mathcal{W}_\beta(\alpha)|\beta\rangle,$$

where we have used the Verlinde formula. Since we've already know $S_{0\beta}$ from previous discussion, then knowing $\mathcal{W}_\beta(\alpha) = \langle\beta|\mathcal{O}^A_\alpha|\beta\rangle$ would allow us to find the full $S_{\alpha\beta}$.

To relate $\langle\beta|\mathcal{O}^A_\alpha|\beta\rangle$ to the geometry of $\mathcal{M}$, we must again use 3d-3d correspondence. It is natural to interpret anyons in $T[\mathcal{M}]$ as M2 branes wrapping around 1-cycles of the internal manifold $\mathcal{M}$. Hence, the anyon α in $T[\mathcal{M}]$ naturally correspond to the loop operators (that is, the Wilson lines) in the $SL(2,\mathbb{C})$ CS theory over $\mathcal{M}$. The Wilson lines are labelled by $a \in \pi_1(\mathcal{M})$ and an irreducible representation R. In the $\hbar \to 0$ limit, we focus on the flat connections $\mathcal{A}_\alpha$, which are equivalent to $SU(2)$ or $SL(2,\mathbb{R})$ connections. The irrep is determined by the number n of $M2$ brane via $R = \mathrm{Sym}^n\Box$.

On the other hand, we can relate the flat connection $\mathcal{A}_\alpha$ as a group homomorphism ρ_α from $\pi_1(\mathcal{M})$ to $SL(2,\mathbb{R})$ or $SU(2)$. Hence, we have the correspondence between anyon α in $T(\mathcal{M})$ and the loop operators

$$\bigotimes_\kappa \left(a^{(\kappa)}_\alpha, R^{(\kappa)}_\alpha\right)$$

where κ runs over linear independent cycles of Σ_g.

This correspondence allows us to identify the vev of $\mathcal{O}^A_\alpha$ at the vacuum $|\beta\rangle$, $\mathcal{W}_\beta(\alpha)$, as the vev of loop operators $\bigotimes_\kappa \left(a^{(\kappa)}_\alpha, R^{(\kappa)}_\alpha\right)$ at the irreducible flat connection ρ_β. Then we have

$$W_\beta(\alpha) = \prod_\kappa \mathrm{Tr}_{R^{(\kappa)}_\alpha}\left(\rho_\beta(a^{(\kappa)}_\alpha)\right).$$

References

[1] G. Y. Cho, D. Gang & H.-C. Kim, *"M-theoretic Genesis of Topological Phases"*, JHEP **2011**, 115 (2020), `arXiv:2007.01532 [hep-th]`.

[2] N. A. Nekrasov & S. L. Shatashvili, *"Supersymmetric vacua and Bethe ansatz"*, Nucl. Phys. B Proc. Suppl. **192-193**, 91 (2009), `arXiv:0901.4744 [hep-th]`.

[3] C. Closset, H. Kim & B. Willett, *"Supersymmetric partition functions and the three-dimensional A-twist"*, JHEP **1703**, 074 (2017), `arXiv:1701.03171 [hep-th]`.

[4] M. Blau & G. Thompson, *"Chern-Simons theory on S1-bundles: Abelianisation and q-deformed Yang-Mills theory"*, JHEP **0605**, 003 (2006), `hep-th/0601068`.