Chiral Central Charge: A Journey from Conformal Field Theory to Topological Phases

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In this paper, we will present a journey about chiral central charge from conformal field theory (CFT) to topological theory of phases. We will first show that the chiral central charge is more about how the system respond to the property of spacetime manifold via a example of modular transformation in both CFT and an anyon model without CFT. Then we will present a purely topological formalism in terms of topological partition functions of gravitational Chern-Simons (CS) term, which can relate the chiral central charge to other quantities about topological phases.

INTRODUCTION

It's very remarkable that any system with non-zero Chern number has gapless modes[1]. These modes are *chiral*, meaning that they propagrate in a certain direction.

The first hint of its relation with topology arised from Hall conductance. In a two-dimensional system of free fermions, when the Fermi energy lies in the j-th energy gap, the Hall conductance is

$$\sigma_{xy}^{\text{edge}} = \sum_{l=1}^{j} \sigma_{xy}^{l,\text{edge}} = -\frac{e^2}{h}\nu(C(\mu_j)), \tag{1}$$

where $\nu(C(\mu_j))$ is the winding number of the loop arond the j-th hole in the Riemann surface of the Bloch function under some rational flux $\phi = p/q$[2]. In fact, the edge modes consist of both left-moving and right-moving modes:

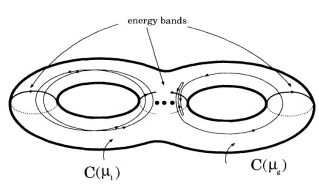

FIG. 1: Riemann surface of the Bloch function under some rational flux $\phi = p/q$. $C(\mu_j)$ is a loop of zero point of wave function $\Psi_q(z)$ around the j-th hole.

$$\nu_{\text{edge}} := (\#\text{ of left-movers} - \#\text{ of right-movers}) = \nu. \tag{2}$$

The chiral edge modes carry energy. It can be derived from CFT that the energy current along the edge in the left direction is

$$I = \frac{\pi}{12}c_- T^2, \tag{3}$$

where T is the temperature and assumed to be much smaller than the energy gap in the bulk[3]. c_- is the chiral central charge:

$$c_- = c - \bar{c}, \tag{4}$$

where c and $\bar{c}$ are the Virasoro central charges. It's related to the edge moded by

$$c_- = \frac{\nu}{2}, \tag{5}$$

which keeps invariant under the change of some conditions in the edge. This indicates that the edge energy currect is a property of the bulk groundstate.

From the introduction above, it's quite natural to believe there must be some links between the chiral central charge in CFT formalism and the winding number or Chern number in topological point of view. In the rest of this paper, we will show that, indeed, the concept of chiral central charge contains information of topological phases and should be understood beyond the scope of CFT.

MODULAR TRANSFORMATIONS IN CONFORMAL FIELD THEORY AND BEYOND

In this section, we will first briefly introduce modular transformation in CFT. Then we will demostrate the correspondence between the characters in CFT and the partition function beyond CFT under the Dehn twist. Such a correspondence indicates the chiral central charge is a more general concept which is related to spacetime manifolds, and can therefore go beyond CFT formalism.

Modular Transformations

In the CFT derivation of the edge current, the chiral central charge gets involved in the results when we perform some conformal transformation to the original Riemann surface[3], and it's related to the calculation of edge energy current by calculating the energy stress tensor. Let's consider a specific example, say a torus, to which we can perform a modular transformation.

The modular transformation is defined as follows[4]: Consider two oreinted cycles $C(\tau)$ and $C(x)$ on the torus. $C(\tau)$ represent a loop parameterized by Euclidean time τ, with direction of time evolution generated by L_0, and $C(x)$ connects points of the equal time. Let $[\phi_i]$ be a representation of a primary field ϕ_i, then the character χ_i of the chiral symmetry algebra is defined by

$$\chi_i = \mathrm{tr}_{[\phi_i]}\left(q^{L_0+\epsilon}\right), \quad (6)$$

where $q = e^{2\pi i\tau}$ and $\epsilon = -c_-/24$. τ is the modular paramter of the torus. Then the actions of modular group are generated by

$$T : \tau \to \tau + 1, \quad S : \tau \to -\frac{1}{\tau}. \quad (7)$$

Then the corresponding transformaiton of the character χ_i is

$$T : \chi_i \to e^{2\pi i(h_i+\epsilon)}\chi_i, \quad S : \chi_i \to \mathcal{S}_i^{\ j}\chi_j, \quad (8)$$

where h_i is the conformal dimension of the primary field ϕ_i and $\mathcal{S}_i^{\ j}$ is a unitary matrix, whose square $\mathcal{S}^2$ inverts the time direction and transforms χ_i to its conjugate representation $\mathcal{S}^2 = C : \chi_i \to \chi_i^*$.

Correspondence between Characters and Partition Functions under Mocular Transformation

Consider the partition function $Z = \mathrm{tr}(e^{-\beta H})$ of an anyon model on a disk with circumfence L, with time period $\beta = 1/T$. We can then make the disk into a manifold $M = D^2 \times S^1$, so that the edge is now a torus. Assume the energy gap in the bulk is much larger than the temperature and act on the system with a time-like Wilson loop operator $W_a(\ell)$, then the partition function is[1]

$$Z_a = \chi_a(1/w) = S_a^{\ b}\chi_b(w) \sim S_a^0\chi_0(w), \quad w \to \infty, \quad (9)$$

where $w = iLT/v$ and the last equality is due to the fact that the characters of excitations are exponentially smaller than the vacuum character.

Now consider Dehn twist. The transformation of the vacuum character under the Dehn twist can be generalized to be

$$\chi_0(w+1) = e^{-2\pi ic_-/24}\chi_0(w), \quad (10)$$

where the subscript "0" stands for vacuum. Then in the thermodynamic limit, the transformation of partition function under the Dehn twist is

$$Z_a' = e^{-2\pi ic_-/24}Z_a, \quad (11)$$

Finally, we can conclude that the characters in CFT corresponds to a linear combinations of the partition function:

$$\chi_a(w) \leftrightarrow \tilde{Z}_a = s_a^{\ \bar{b}}Z_b, \quad (12)$$

where s is a topological S-matrix defined by anyonic brading. They all have the transformation property

$$\tilde{Z}_a' = e^{-2\pi ic_-/24}\theta_a\tilde{Z}_a \leftrightarrow \chi_a(w+1) = e^{-2\pi ic_-/24}\theta_a\chi_a, \quad (13)$$

where θ_a is the topological spin of a.[1] Therefore, we can conclude that the chiral central charge gets involved in the transformation of the partition function under the modular transformation of spacetime manifold.

From this section, we can have a sense that the chiral central charge should be related to the topology of the spacetime manifold via partition functions.

CHIRAL CENTRAL CHARGE AND TOPOLOGICAL PHASES

In this section, we will present the relation between the chiral central charge and the topological phases. To follow the previous discussions of partition function, we will use topological partition functions (or path integral) formalism and relate the chiral central charge to other characters of topological phases.

Topological Partition Function and Winding Numbers

There is an amazing reasult that a boundary-gappable topological order is fully characterized by a collection of representations of the mapping class groups $\mathrm{MCG}(M_D)$ for various spatial topologies[5]. The $\mathrm{MCG}(M_D)$ group is defined in terms of the orientation preserving homeomorphism group: $\mathrm{MCG}(M_D) \equiv \pi_0[G_{\mathrm{homeo}}(M_D)]$.

In general, a partition function on a closed D dimensional spacetime manifold M_D is defined as

$$Z(M_D) = e^{-c_D L^D - c_{D-1}L^{D-1} - \cdots - c_0 L^0 - c_{-1}L^{-1} - \cdots}, \quad (14)$$

where L is the linear size of M_D. Suppose the groundstate doesn't contain point-like, string-like, ect defects, then $c_d = 0$ for $d = 1, 2, \cdots, D-1$. The partition function is now

$$Z_t(M_D) = \lim_{L\to\infty}\frac{Z(M_D)}{e^{-c_D L^D}}, \quad (15)$$

which is volume-indenpendent and topological.

Let $\mathcal{M}_D$ be the moduli space of M_D, then the non-zero $Z_t(\cdot)$ forms a map

$$Z_t : \mathcal{M}_D \to \mathbb{C} - \{0\} \sim U(1). \quad (16)$$

Then because $\pi_1(\mathcal{M}_D) = \mathrm{MCG}(M_D)$, the winding number is a group homomorphism

$$\mathrm{MCG}(M_D) \to \mathbb{Z} = \pi_1(U(1)), \quad (17)$$

which can be realized by the topological partition function $Z_t(M_D)$ of the gravitational CS terms ω_D[5–7]:

$$Z_t(M_D) \sim \exp\left(i\int_{M_D}\omega_D\right). \quad (18)$$

For a spacetime of dimension $D = 4n+3$, the winding number ν is then given by

$$\nu = \int_{M_D \times S^1} d\omega_D. \tag{19}$$

The calculation of the edge energy currect does include such a gravitational CS term[6], and chiral central charge $c_- = \text{sgn}(K)$ gets involved as a coefficient factor[7]. Therefore, we can see it's related to the winding number. In the next section, we will show its relation to Chern number.

Information of Topological Phases

After we present the topological partition function and its relation to the winding number, we can then obtain the information out of the partition function and find some interesting relation between chiral central charge and topological phases.

Consider a d dimensional closed space Σ_d, in which the Hamiltonian is well-defined and gapped. The spacetime M_D is $\Sigma_d \times S^1$, with $D = d+1$. The topological partition function is

$$Z_t(M_D) \sim \exp\left(i \int_{M_D} \omega\right) = \exp\left(i \int_{V_D} d\omega\right), \tag{20}$$

where $V_D = \Sigma_d \times B$ and $\partial V_D = \Sigma_d \times S^1$. Notice $V_D = \Sigma_d \times B$ can be considered as a fiber bundle with the space Σ_d and base manifold B and $d\omega$ is given by combinations of Pontryagin class $d\omega = P = \kappa^{n_1 n_2 \cdots} P_{n_1 n_2 \cdots}$, then the integral in Eq.(20) includes Chern number

$$\int_B C = \int_{V_D} P_{n_1 n_2 \cdots} \tag{21}$$

of a complex line bundle on B[8]. It can be further shown that

$$D_{\Sigma_d} \int_{V_D} d\omega \in \mathbb{Z}, \tag{22}$$

where $D_{\Sigma_d} = |Z_t M_D|$ is the groundstate degeneracy. This result gives a constrain between $\int d\omega$ and D_{Σ_d} and relates the chiral central charge to Chern numbers.

For example, consider a $D = 2n+1$ theory, say $d = 2n = 2$, we can conclude from Eq.(22) that

$$\frac{c_-}{24} D_g \int_{\Sigma_2 \times B^2} p_1 = \int_{B^2} C \in \mathbb{Z}, \tag{23}$$

where p_1 is the first Pontryagin class and the result of its integration is ± 12. Therefore, we conclude that for a $2d$ bosonic topological orders, the chiral central charge of the edge state is quiantized as $cD_g/2 \in \mathbb{Z}$, where D_g is the groundstate degeneracy on genus-g space. There is also a similar result for fermionic topological order[5].

With these results, the chiral central charge can provide lots of information. Here are two examples[8]:

- If the central charge of a bosonic quantum Hall state is 1, then the groundstate degeneracy for $g \geq 3$ space must be even.
- Given the chiral central charge, we can obtain the statistic of the topological excitations:

$$e^{2\pi i c_-/8} = \frac{\sum_\alpha d_\alpha^2 e^{i\theta_\alpha}}{\sqrt{\sum_\alpha d_\alpha^2}}, \tag{24}$$

  where α labels the particle-like excitations. θ_α is the statistic angle and d_α is the quantum dimension.

We can see that, by manipulating the topological partition functions, we could obtain many conclusions of topological phases via chiral central charge. Even more, there is a conjuecture, claiming that given the chiral central charge and representations of the mapping class groups for all genus-g surface, one can fully characterize $2d$ topological orders[9].

By now, we finished our journey about chiral central charge from CFT to topological phases. The formalism we presented in this paper is mainly in terms of partition functions. Actually, there is another formalism, namely Hamiltonian formalism, in which we can also define chiral central charge by constructing 2-current from gapped local Hamiltonian $H = \sum_j H_j$. Starting from this definition, we can also see the relation between chiral central charge and other topological quantities such as Chern numbers[1, 5].

SUMMRIZE

In this paper, we started from a specific example – modular transformation to show that the chiral central charge is not a mere concept in CFT, but more importantly, a topology-related quantity. From the example of modular transformation, we can see that we could find the role of the central charge in the description of the system via partition functions. This formalism, as we presented in the last section, can be applied in topological theory. Then by playing with the partition functions, we can conclude lots of relations between the chiral central charge and other quantities of topological phases.

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