Four-Manifolds, Donaldson Theory and Supersymmetric Topological Field Theories

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In this short term paper, I will provide a birds-eye view of the deep and fascinating connection between Donaldson theory on smooth 4-manifolds and supersymmetric Yang-Mills theories, which is one of the many beautiful offsprings from the happy marriage of physics and math. Lots of details are glossed over and many facts are stated without proof, so reference to original literature is highly encouraged.

INTRODUCTION

In mathematics, classification of topological manifolds in four dimensions had been a difficult problem. Many of the techniques used for manifolds in higher dimensions fail for 4-manifolds. Freedman's results in 1982 helped to classify topological 4-manifolds up to homeomorphism (continuous bijection with continuous inverse) based on intersection forms [3]. However, Freedman's results did not solve the classification of topological smooth 4-manifolds up to diffeomorphism (bijective maps and inverses have to be smooth). This problem was solved by Donaldson in 1983 by studying the moduli space of SU(2) instantons of Yang-Mills theory defined on smooth 4-manifolds [2]. Later in 1988, Witten considered topological supersymmetric YM theory with matter fields added, and reformulated Donaldson invariants in terms of correlation functions [6]. The reformulation in correlation functions was not a useful way to actually calculate the invariants. It was not until 1994, when Seiberg and Witten solved the $\mathcal{N} = 2$ supersymmetric YM theory by finding the abelian dual theory of monopoles [5], that Witten was able to calculate the Donaldson invariants using the U(1) monopole moduli space [7].

The field theory approaches make sense because in both cases the topological properties of the field theories spit out information about the topology of the underlying 4-manifold. This is reminiscent of the famous index theorem, which also played an essential role in the theories of Donaldson and Seiberg-Witten.

In writing this short paper, I also benefited a lot from review lectures by Dijkgraaf [1] and those by Marino [4].

DONALDSON THEORY

Donaldson theory studies the non-abelian SU(2) Yang-Mills theory defined on a 4-manifold X, on which we can define the principal bundle P with section A as the SU(2) gauge field. F is the corresponding curvature.

Moduli Space for SU(2) Instantons

Given a metric g on X, the Hodge $*$ operation can be defined and it satisfies $*^2 = 1$, which means the operator has eigenvalues ± 1, with corresponding 2-form eigenvectors: $*F_\pm = \pm F_\pm$. For any F, we can construct these eigenvectors to be $F_\pm = F \pm *F$. Now consider the YM theory

$$\begin{aligned} S_{YM} &= \frac{1}{2}\int_X \mathrm{tr} F \wedge *F \\ &= \frac{1}{4}\int_X \mathrm{tr}(F_\pm)^2 \mp 2\mathrm{tr} F * F \\ &\geq \frac{1}{4}\int_X \mp 2\mathrm{tr} F * F = 8\pi^2 |\mathcal{I}|, \end{aligned} \tag{0.1}$$

where $\mathcal{I}$ is the instanton number. The classical minimum of the YM action is achieved when $F_+ = 0$ (i.e. $F = - * F$) or $F_- = 0$ (i.e. $F = *$F). These solutions are naturally called anti-self-dual (ASD) and self-dual (SD). They are also called instanton solutions because the action is completely determined by the instanton number at these solutions. With out loss of generality, we will just choose to focus on the ASD instantons, i.e. $F_+ = 0$.

We have to note that these solutions may not be completely inequivalent, giving the possibility of relating different field configurations through gauge transformations. Thus, the actually ASD solution space, or the *moduli space*, is given by $\mathcal{M} = \{[A] \in \mathcal{A}/\mathcal{G} | F_+(A) = 0\}$, where $\mathcal{A}$ is the space of connections and $\mathcal{G}$ is the space of gauge transformations.

Donaldson Invariants

The main idea of Donaldson theory is to relate the properties of the moduli space $\mathcal{M}$ to the topology of the underlying smooth 4-manifold X. Firstly, we need a map to relate the two very different spaces. Given a second Chern class c_2 (a 4-form) on $X \times \mathcal{M}$, then for any k-form $\alpha \in H^k(X)$, we can define

$$\hat{\alpha} \equiv \int_X \alpha \wedge c_2, \tag{0.2}$$

which is a k-form in $H^k(\mathcal{M})$. This is the map that relates $H^k(X)$ and $H^k(\mathcal{M})$. Under the assumption that

X is simply connected, the only non-trivial cohomology classes are the 2-forms $v \in H^2(X)$ and the volume 4-form $\lambda \in H^4(X)$. With these ingredients, we can finally define the Donaldson polynomials corresponding to instanton number n as

$$D_{\mathcal{I}}(v, \lambda) = \int_{\mathcal{M}_{\mathcal{I}}} \exp(\hat{v} + \hat{\lambda}), \tag{0.3}$$

which, after Taylor expansion, is a polynomial containing finite number of terms with total degree matching the dimension of $\mathcal{M}_{\mathcal{I}}$, the moduli space corresponding to instanton number $\mathcal{I}$. It was proven by Donaldson that these polynomials are diffeomorphism invariants under certain conditions ($b_2^+ > 1$). Here $b_2^\pm$ is the number of positive/negative eigenvalues of the intersection matrix on X. This condition make it possible to avoid singularities on the moduli space by deforming the metric. (Note that the moduli space is metric dependent, even though the topological information obtained from it does not depend on the choice of metric)

We can further introduce the generating function for these invariants by summation over all instanton numbers:

$$D(v, \lambda) = \sum_{\mathcal{I} \geq 0} D_{\mathcal{I}}(v, \lambda). \tag{0.4}$$

SEIBERG-WITTEN THEORY

Dual Abelian Theory in Monopoles

Witten managed to reformulate the Donaldson invariants using correlation functions in his topological supersymmetric YM theory, but the correlation functions are hard to evaluate without knowing the ground state of the theory. Later Seiberg and Witten solved the theory by mapping the original non-abelian theory to an dual abelian theory in terms of monopoles.

Now the main players in this dual abelian theory are the abelian gauge connection $A \in \mathcal{A}$ and a charged spinor field M, following the notations in the original literature. Here A is a section of a complex line bundle L and M is a section of the bundle $S^+ \otimes L$, where S^+ is a positive spin bundle of rank 2 defined on X. Similarly S^- is the negative counterpart of rank 2 and the complex conjugate spinor $\bar{M}$ would be a section of $S^- \otimes L$. X allows spin structures when its second Stiefel-Whitney class $w_2(X)$ vanishes, which we assume is the case here. Then the U(1) monopole equations are given by

$$\begin{aligned} F_{ij}^+ &= -\frac{i}{4} \bar{M} [\Gamma_i, \Gamma_j] M, \\ \Gamma^i D_i M &= 0, \end{aligned} \tag{0.5}$$

where Γ_i are the Clifford matrices satisfying $\{\Gamma_i, \Gamma_j\} = 2g_{ij}$. Similar to the moduli space of the SU(2) instantons, the U(1) monopole moduli space is now given by

$$\mathcal{M}_x = \{A \in \mathcal{A}, M \in s(S^+ \otimes L) | Eq.(0.5)\}/\mathcal{G}, \tag{0.6}$$

where the $x \in H^2(X)$ can be characterized by the first Chern class with Chern number being the monopole number and $\mathcal{G}$ again is the space of gauge transformations.

Differences in Moduli Spaces: $\mathcal{M}_I$ v.s. $\mathcal{M}_x$

Now it's a good place to mention some of the main differences between the Donaldson SU(2) instanton moduli space $\mathcal{M}_{\mathcal{I}}$ and the Seiberg-Witten U(1) monopole moduli space $\mathcal{M}_x$.

- $\mathcal{M}_x$ is only non-empty for finite number of $x \in H^2(X)$, whereas for $\mathcal{M}_{\mathcal{I}}$ the instanton number $\mathcal{I}$ can be arbitrarily large.
- $\mathcal{M}_x$ is compact, whereas $\mathcal{M}_{\mathcal{I}}$ is non-compact. Because of this, the integration over $\mathcal{M}_{\mathcal{I}}$ in the defintion of Donaldson invariants has to be carefully justified, whereas the integration over $\mathcal{M}_x$ for the SW invariants is manifestly well-defined.

Seiberg-Witten Invariants

In analogy to Donaldson theory, we need a map between cohomology groups of X and that of $\mathcal{M}$. The map is now defined using the first Chern class c_1 on $X \times \mathcal{M}$. For any $\alpha \in H^k(X)$,

$$\hat{\alpha} \equiv \int_X \alpha \wedge c_1, \tag{0.7}$$

from which we see that a k-form on X is mapped to a $k-2$-form on $\mathcal{M}$. More precisely, for a 2-form v, $\hat{v}$ is just a number; for a 4-form λ, $\hat{\lambda}$ is a 2-form on $\mathcal{M}$. Now we are ready to define the Seiberg-Witten invariants corresponding to a particular monopole number to be

$$\mathrm{SW}_x = \int_{\mathcal{M}_x} \exp(\hat{\lambda}) \tag{0.8}$$

In order to relate the Donaldson invariants to the Seiberg-Witten invariants, we have to define a new generating function for Donaldson polynomials on 4-manifolds of *simple type*. A 4-manifold is of simple type if the generating function Eq. (0.4) is annihilated by $\frac{\partial^2}{\partial \lambda^2} - 4$, i.e.

$$\frac{\partial^2}{\partial (2\lambda)^2} D(v, \lambda) = D(v, \lambda). \tag{0.9}$$

Because of this, only the zeroth order and first order derivatives carry non-trivial information. Thus, we can define a new generating function as the following

$$\mathcal{D}(v) = D(v, 0) + \frac{1}{2} \frac{\partial}{\partial \lambda} D(v, 0). \tag{0.10}$$

It can be shown that for X of simple type and $b_2^+ > 1$,

$$\mathcal{D}(v) = 2^{(7\chi+11\sigma)/4+2} \mathrm{e}^{v^2/2} \sum_x \mathrm{e}^{v\cdot x} \mathrm{SW}_x. \qquad (0.11)$$

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