Construction of abelian higher gauge theories and their topological entanglement entropy

Zhehao Zhang[1]

[1] *Department of Physics, University of California at San Diego, La Jolla, CA 92093*

This paper reviews the construction for abelian higher gauge theories using homological algebra and a general formula to calculate entanglement entropy (EE) and topological entanglement entropy (TEE).

1. INTRODUCTION

Nowadays, two dimensional topological order and TEE[1] are relatively well understood. If we want to study topological order in dimension larger than 2, do we expect similar behaviors of the TEE compared to TEE in 2D[2]? To gain more insights on topological order in higher dimensions, people have studied the general structure of EE and TEE of abelian higher gauge theories on lattice. Those theories can be interpreted as higher gauge generalizations of the Toric Code.

This review paper is mainly based on ref. [3] and [4]. The content is organized as follows. In section 2, we discuss the construction of lattice models that come from higher gauge theories using homological algebra. In section 3, we summarize the general formula for EE and TEE.

2. REVIEW ON ABELIAN HIGHER GAUGE THEORIES

Geometrical content and group theoretic content

We consider a general lattice in d dimension. The lattice K is made of simplices, where we can write it as $K = K_0 \cup K_1 \cup ... \cup K_d$ and K_n is the set of n-dimensional simplices.

Let C_n be the Abelian group freely generated by K_n. If we write the group operation as addition, each group element c in C_n can be written as $c = \sum_{x \in K_n} n(x)x$, where $n(x) \in \mathbb{Z}$.

We have the usual boundary map $\partial_n^C : C_n \to C_{n-1}$. Using the boundary map we can define an chain complex $(C(K), \partial^C)$ to describe the geometrical content of the models

$$0 \to C_d \xrightarrow{\partial_d^C} C_{d-1} \xrightarrow{\partial_{d-1}^C} ... \xrightarrow{\partial_1^C} C_0 \to 0. \tag{1}$$

To describe the higher gauge groups that label the degrees of freedom in the simplicial complex, we define another chain complex (G, ∂^G) of finite Abelian groups,

$$0 \to G_d \xrightarrow{\partial_d^G} G_{d-1} \xrightarrow{\partial_{d-1}^G} ... \xrightarrow{\partial_1^G} G_0 \to 0, \tag{2}$$

where 0 denotes the trivial group and $\partial_n^G : G_n \to G_{n-1}$ are group homomorphisms such that $\partial_p^G \cdot \partial_{p+1}^G = 0$, for any $0 \le p \le d$.

A gauge configuration $f = \{f_n\}_{n=0}^d$ is a sequence of functions such that $f_n : K_n \to G_n$. Because C_n is freely generated by K_n, each map f_n defines a unique group homomorphism $f_n : C_n \to G_n$.

We define $\mathrm{Hom}(C_n, G_n)$ to be the set of all homomorphisms f_n and $\mathrm{Hom}(C_n, G_n)$ is itself an Abelian group if we set

$$(f_n + \tilde{f}_n)(x) = f_n(x) + \tilde{f}_n(x), f_n, \tilde{f}_n \in \mathrm{Hom}(C_n, G_n) \tag{3}$$

We can collect all such $\mathrm{Hom}(C_n, G_n)$ into a single direct sum

$$\mathrm{hom}(C, G)^0 := \bigoplus_{n=0}^{d} \mathrm{Hom}(C_n, G_n). \tag{4}$$

A gauge configuration $f \in \mathrm{hom}(C, G)^0$ can be represented by a collection of maps between $(C(K), \partial^C)$ and (G, ∂^G), see Fig 1.

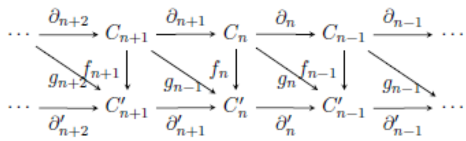

FIG. 1: Example of two elements $f \in \mathrm{hom}(C, G)^0$ and $g \in \mathrm{hom}(C, G)^1$

Generally, we can define

$$\mathrm{hom}(C, G)^p := \bigoplus_{n=0}^{d} \mathrm{Hom}(C_n, G_{n-p}).$$

The abelian groups $\mathrm{hom}(C, G)^p$ give rise to a cochain complex $(\mathrm{hom}(C, G)^\bullet, \delta^\bullet)$

$$... \to \mathrm{hom}(C, G)^p \xrightarrow{\delta^p} \mathrm{hom}(C, G)^{p+1} \to ..., \tag{5}$$

where $\delta^p : \mathrm{hom}(C, G)^p \to \mathrm{hom}(C, G)^{p+1}$ is defined by

$$(\delta^p f)_n := f_{n-1}\partial_n^G - (-1)^p \partial_{n-p}^G f_n, \tag{6}$$

and it can be checked that $\delta^{p+1} \cdot \delta^p = 0$.

We want to define a chain complex that is the dual of equation (5). Since G_n is a finite abelian group, all irreducible unitary representations of G_n forms an abelian group, which we denote as $\hat{G}_n$. We define the dual of $\mathrm{hom}(C,G)^p$ as $\mathrm{hom}(C,G)_p$,

$$\mathrm{hom}(C,G)_p := \bigoplus_{n=0}^{d} \mathrm{Hom}(C_n, \hat{G}_{n-p}). \tag{7}$$

Introducing the dual description allows us to define something analogous to inner product. We define the pairing $\langle\cdot,\cdot\rangle : \mathrm{hom}(C,G)_p \times \mathrm{hom}(C,G)^p \to U(1)$ to be

$$\langle m, f\rangle := \prod_{n=0}^{d} \prod_{x\in K_n} m_n(x)(f_n(x)). \tag{8}$$

Let us define the dual boundary map δ_p : $\mathrm{hom}(C,G)_p \to \mathrm{hom}(C,G)_{p-1}$ to be

$$\langle \delta_p m, f\rangle = \langle m, \delta^{p-1} f\rangle. \tag{9}$$

Evidently, $\delta_p \cdot \delta_{p+1} = 0$ and thus the dual chain complex is given by

$$... \xleftarrow{\delta_{p-1}} \mathrm{hom}(C,G)_{p-1} \xleftarrow{\delta_p} \mathrm{hom}(C,G)_p \leftarrow ... \tag{10}$$

Hilbert space, operators, and Hamiltonian

An orthonormal basis $\{|f\rangle\}$ of the Hilbert space is defined by

$$|f\rangle := \bigotimes_{n} \bigotimes_{x\in K_n} |f_n(x)\rangle\,, f \in \mathrm{hom}(C,G)^0. \tag{11}$$

We can define generalized gauge transformation operators A_t and holonomy measure operators B_m. For $t \in \mathrm{hom}(C,G)^{-1}$ and $m \in \mathrm{hom}(C,G)_1$,

$$A_t\,|f\rangle := \left|f + \delta^{-1} t\right\rangle, \quad B_m\,|f\rangle := \langle m, \delta^0 f\rangle\,|f\rangle\,. \tag{12}$$

However, we notice that A_t and B_m act on the entire lattice, so for the purpose to define a Hamiltonian, we need to define localized gauge transformation operators and localized holonomy measure operators.

Let $x \in K_n$, $g \in G_{n+1}$ and $r \in \hat{G}_{n-1}$. We define the local maps $\hat{e}[n,x,r] \in \mathrm{hom}(C,G)_1$ and $e[n,x,g] \in \mathrm{hom}(C,G)^{-1}$ by

$$e[n,x,g](y) := \begin{cases} g, & \text{if } y = x \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

$$\hat{e}[n,x,r](f) := r(f_n(x)), \tag{14}$$

where $y \in K$, and $f \in \mathrm{hom}(C,G)^p$. We define local gauge projector $A_{n,x}$ and local holonomy projector $B_{n,x}$ as:

$$A_{n,x} = \frac{1}{|G_{n+1}|} \sum_{g\in G_{n+1}} A_{e[n,x,g]}, \tag{15}$$

$$B_{n,x} = \frac{1}{|G_{n-1}|} \sum_{r\in \hat{G}_{n-1}} B_{\hat{e}[n,x,r]}. \tag{16}$$

Using the localized projectors defined above, the Hamiltonian is written as

$$H = -\sum_{n=0}^{d} \sum_{x\in K_n} A_{n,x} - \sum_{n=0}^{d} \sum_{x\in K_n} B_{n,x} \tag{17}$$

It is straightforward to verify that $A_{n,x}$ and $B_{n,x}$ are commuting projectors, so a state in the groundstate has eigenvalue 1 for all $A_{n,x}$ and $B_{n,x}$.

Let's define another two projectors to characterize the groundstate Hilbert space $\mathcal{H}_0$:

$$\mathcal{A}_0 = \frac{1}{|\,\mathrm{hom}(C,G)^{-1}|} \sum_{t\in \mathrm{hom}(C,G)^{-1}} A_t \tag{18}$$

$$\mathcal{B}_0 = \frac{1}{|\,\mathrm{hom}(C,G)_1|} \sum_{n\in \mathrm{hom}(C,G)_1} B_m, \tag{19}$$

Projector$\mathcal{A}_0$ maps any state into a normalized sum of gauge equivalent state. Projector $\mathcal{B}_0$ gives eigenvalue 1 for state $|f\rangle$, only if $f \in \ker\,(\delta^0)$, which is also called flat holonomy (See ref. [3] for a detailed explanation).

The projector that projects on to the groundstate subspace $\mathcal{H}_0$ is $\Pi_0 := \mathcal{A}_0\mathcal{B}_0$. The dimension GSD of $\mathcal{H}_0$ is given by the number of flat states $|\ker\,(\delta^0)|$, modulo the gauge equivalence $|\mathrm{Im}\,(\delta^{-1})|$, that is

$$GSD = \frac{|\ker\,(\delta^0)|}{|\mathrm{Im}\,(\delta^{-1})|} = |H^0(C,G)|. \tag{20}$$

3. ENTANGLEMENT ENTROPY AND TOPOLOGICAL ENTANGLEMENT ENTROPY IN ABELIAN HIGHER GAUGE THEORIES

Bipartition of the geometrical chain complex

In order to compute the entanglement entropy of a region A in the lattice, we need to define the bipartition of the lattice into regions A and B. We divide each n-simplicies into the form $K_n = K_{n,A} \cup K_{n,B}$. The subcomplex K_A is given by $K_A = \bigcup_{n=0}^{d} K_{n,A}$.

Similarly, we have a chain complex $(C(K_A), \partial_A^C)$ associated to the subcomplex K_A. Homomorphisms between $(C(K_A), \partial_A^C)$ and the same chain complex of abelian groups give rise to the groups

$$\mathrm{hom}(C_A, G)^p := \bigoplus_{n} \mathrm{Hom}(C_{n,A}, G_{n-p}) \tag{21}$$

Reduced density matrix

We consider a quantum state $|\psi\rangle$, which is the equal weight superposition of all the groundstates. Its density matrix is given by,

$$\rho := \frac{\Pi_0}{\mathrm{tr}\Pi_0} = \frac{\Pi_0}{GSD} \tag{22}$$

Using the expression in equation () and rewriting the summation in $\mathcal{A}_0$ and $\mathcal{B}_0$ to only sum over elements that act non-trivially on the states. The density matrix can be written as

$$\rho = \frac{1}{GSD}\frac{1}{|\mathrm{Im}\,(\delta^{-1})||\mathrm{Im}\,(\delta_1)|}\left(\sum_{[t]} A_t\right)\left(\sum_{[m]} B_m\right), \tag{23}$$

where $[t] \in \frac{\hom(C,G)^{-1}}{\ker\ (\delta^{-1})}$ and $[m] \in \frac{\hom(C,G)_1}{\ker\ (\delta_1)}$.

Now we are ready to calculate the reduced density matrix for A,

$$\rho_A = \sum_i \langle b_i|\,\rho\,|b_i\rangle\,, \tag{24}$$

where $\{|b_i\rangle\}$ is the basis of the Hilbert space over region B.

When evaluating the summation in ρ_A, there is a subtlety for gauge transformation operators labeled by elements that lie at the boundary of A, because they do not act exclusively on region A[3]. So we only consider gauge transformation in the interior of A, which we call $\tilde{A}$. Let $K_{n,\tilde{A}} = \{x \in K_{n,A}| x \cap \partial A = \emptyset\}$ and $\tilde{A} := \bigcup_{n=0}^{d} K_{n,\tilde{A}}$.

The final expression for ρ_A is

$$\rho_A = \frac{1}{\dim(\mathcal{H}_A)}\left(\sum_{p,q} A_p B_q\right), \tag{25}$$

where where $p \in \frac{\hom(C_{\tilde{A}},G)^{-1}}{\ker\ (\delta_{\tilde{A}}^{-1})}$ and $q \in \frac{\hom(C_A,G)_1}{\ker\ (\delta_1|_A)}$.

Entanglement entropy

Using the expression

$$\rho_A^2 = \frac{|\mathrm{Im}\,(\delta_A^0)||\mathrm{Im}\,(\delta_{\tilde{A}}^{-1})|}{\dim(\mathcal{H}_A)}\rho_A = \lambda\rho_A. \tag{26}$$

The EE of region A is given by

$$S_A = -\mathrm{Tr}\ (\rho_A \log(\rho_A)) = \log(\frac{1}{\lambda}), \tag{27}$$

where we evaluate the logarithm of ρ_A by series expansion and we use Tr $(\rho_A) = 1$.

Using the expression,

$$\dim(\mathcal{H}_A) = |\hom(C_A,G)^0| = |\ker\ (\delta_A^0)|\ |\mathrm{Im}\,(\delta_A^0)|$$

we can relate S_A to the ground state degeneracy of $\mathcal{H}_{\tilde{A}}$, that the EE is given by:

$$S_A = \log\left(\frac{|\ker\ (\delta_A^0)|}{|\mathrm{Im}\,(\delta_{\tilde{A}}^{-1})|}\right) = \log(GSD_{\tilde{A}}). \tag{28}$$

Topological entanglement entropy

Notice that the quantity $GSD_{\tilde{A}}$ has relation to both region A and the boundary ∂A. If we want to extract subleading topological terms from S_A, we would have topological contribution to the EE from topological invariants of region A and topological invariants of the entangling surface ∂A. Thus, for higher gauge theories, the TEE depends on the Betti numbers of A and ∂A.

With some detail calculations (See ref. [3]), one finds that S_A can be written as

$$S_A = S_{\partial A} + S_{Topo}, \tag{29}$$

where

$$S_{\partial A} = \sum_{n=0}^{d-1}\sum_{p=1}^{d}(-1)^{(p+1)}|K_{n,\partial A}|\log(|G_{n+p}|) \tag{30}$$

corresponds to the area law term and

$$\begin{aligned} S_{\mathrm{Topo}} = &\sum_{n=0}^{d}\log(|H^n(C_A, H_n(G))|) \\ &+ \sum_{n=0}^{d-1}\sum_{p=1}^{d}(-1)^p\log(|H^n(C_{\partial A}, H_{n+p}(G))|) \end{aligned} \tag{31}$$

is the topological entanglement entropy. We can see that S_{Topo} explicitly depends on the topology of A and ∂A.

4. CONCLUSION

To sum up, homological algebra is a natural structure to study abelian higher gauge theories on lattice. The formalism allows us to systematize the computation of ground state degeneracies, EE, and TEE.

In section 2, we explicitly calculate the GSD of this kind of models and show that it is equal to the dimension of the homology group $H^0(C,G)$.

We started to calculate the EE by first defining the density matrix of ground state. To obtain the reduced density matrix we considered a general bipartition of the simplicial complex K into a subcomplex A and its complement B. The reduced density matrix ρ_A included operators that exclusively act on region A. Using the expression for ρ_A, we computed the EE of region A and it can be interpreted as the ground state degeneracy of region A. Then we further divided S_A into two terms: one being the area law term, a term depending on the geometry of the entangling surface ∂A, and the other being the TEE, a term depending on the topological properties of both A and ∂A.

Acknowledgements
I would like to thank Professor McGreevy for teaching us Algebraic Topology this quarter. This amazing course enables me to appreciate the beauty of topology in physics.

[1] A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006). 1

[2] T. Grover, A. M. Turner, and A. Vishwanath, Physical Review B **84**, 195120 (2011). 1

[3] J. Ibieta-Jimenez, M. Petrucci, L. Q. Xavier, and P. Teotonio-Sobrinho, Journal of High Energy Physics **2020**, 1 (2020). 1, 2, 3

[4] R. C. de Almeida, J. Ibieta-Jimenez, J. L. Espiro, and P. Teotonio-Sobrinho, arXiv preprint arXiv:1711.04186 (2017). 1