Jacobi Forms and N=2 SCFT

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We briefly review the connection between Jacobi form and elliptic genus in N=2 superconformal field theory.

INTRODUCTION

In N=1 supersymmetric theories we can define Witten index which contains information about supersymmetric ground state and is robust to perturbation of the Hamiltonian. Similarly in N=2 supersymmetric theories we can define elliptic genus which contains more information than Witten index and is also robust to perturbation. What makes elliptic genus interesting is that it has non-trivial transformation properties under modular and elliptic transformation such that in some cases it can be computed exactly. This kind of object has been studied by mathematicians and is called Jacobi form.

N=2 SCA AND WITTEN INDEX

N=0,1,2,4 superconformal algebra often occur in the study of string theory on various compact manifolds. We are particularly interested in N=2 SCFT whose target space is a Calabi-Yau manifold of dimension 4m. On top of the bosonic energy-momentum tensor T(z), there are two more fermionic currents $G_+(z)$, $G_-(z)$ and a U(1) R-symmetry current J(z). The algebra reads

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\
[J_m, J_n] &= \frac{c}{3}m\delta_{m+n,0} \\
[L_n, J_m] &= -mJ_{m+n} \\
[L_n, G_r^{\pm}] &= (\frac{n}{2}-r)G_{r+n}^{\pm} \\
[J_n, G_r^{\pm}] &= \pm G_{r+n}^{\pm} \\
\{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}(r^2-\frac{1}{4})\delta_{r+s,0}
\end{aligned} \tag{1}$$

There are two possible periodic conditions for the fermions

$$\begin{aligned}
2r &= 0 \quad mod \quad 2, \quad for \quad R \quad sector \\
2r &= 1 \quad mod \quad 2, \quad for \quad NS \quad sector
\end{aligned} \tag{2}$$

We require the ground states to be annihilated by all the positive modes:

$$L_n|\phi>= J_m|\phi>= G_r^{\pm}|\phi>= 0 \tag{3}$$

for all $m, n > 0$, $r \geq 0$.
Since $\{G_0^+, G_0^+\} = 0$, we can define the cohomology of the G_0^+ operator. The fact that $\{G_0^+, (G_0^+)^+\} = L_0 - \frac{c}{24}$ implies the Ramond ground states have the interpretation as the harmonic representatives in the cohomology. Now we can define the Witten index:

$$\begin{aligned}
WI(\tau,\bar{\tau}) &= Tr((-1^{\tilde{J}_0+J_0})\bar{q}^{\tilde{L_0}-\frac{\tilde{c}}{24}}q^{L_0-\frac{c}{24}}) \\
q &= e^{2\pi i\tau}
\end{aligned} \tag{4}$$

It is clear that Witten index only counts states that are Ramond ground states for both the left and right- moving copy of N=2 SCA.

ELLIPTIC GENUS AND JACOBI FORMS

For N=2 SCFT we can gain more information about the theory by computing the elliptic genus defined below:

$$\begin{aligned}
Z(\tau,z) &= Tr((-1^{\tilde{J}_0+J_0})y^{J_0}\bar{q}^{\tilde{L_0}-\frac{\tilde{c}}{24}}q^{L_0-\frac{c}{24}}) \\
y &= e^{2\pi iz}
\end{aligned} \tag{5}$$

SUSY(N=2) pairs all states which are not ground states into pairs with opposite $(-1)^F$.
The elliptic genus provides a compromise between the partition function and the Witten index in the following sense. The former contains a lot more information than the latter which only know about the states that are Ramond ground states with respect to both the left and the right-moving copy of N=2 SCA. The elliptic genus on the other hands contains information of states that are Ramond ground states with respect to only the right-moving copy of N=2 SCA but still has the rigidity of the Witten index which makes it possible to compute for many SCFTs.
Now we can consider transformation properties of elliptic genus under modular and elliptic transformation. Using the usual path integral formalism we can see that $Z(\tau, z)$ should transform "nicely" under modular transformations. In fact, under modular transformation:

$$Z(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = e^{2\pi im\frac{cz^2}{c\tau+d}}Z(\tau,z) \tag{6}$$

2m is the complex dimension of the Calabi-Yau manifold. The elliptic transformation can be motivated by the "spectral flow":

$$\begin{aligned} L_n &\to L_n + \mu J_n + \frac{c}{6}\mu^2\delta_{n,0} \\ J_n &\to J_n + \frac{c}{3}\mu\delta_{n,0} \\ G_r^{\pm} &\to G_{r\pm\mu}^{\pm} \\ \mu &= \frac{1}{2} \\ c &= \frac{m}{6} \end{aligned} \tag{7}$$

This is an isomorphism between R algebra and NS algebra. By doing this twice we can transform $R \to R, NS \to NS$.
We can check that under spectral flow by integer μ

$$\begin{aligned} L_0 &\to L_0 + \mu J_0 + m\mu^2 \\ J_0 &\to J_0 + 2m\mu \\ e^{2\pi i\tau L_0}e^{2\pi i z J_0} &\to e^{2\pi i\tau L_0}e^{2\pi i J_0(z+\mu\tau)}e^{2\pi i\tau m\mu^2}e^{2\pi i m(2\mu\tau)} \end{aligned} \tag{8}$$

This implies that:

$$\begin{aligned} Z(\tau, z+\mu\tau) &= e^{-2\pi i m(\tau\mu^2+2\mu z)}Z(\tau, z) \\ Z(\tau, z+\lambda) &= Z(\tau, z) \\ \lambda &\in Z \end{aligned} \tag{9}$$

The transformation properties under modular and elliptic transformation imply that $Z(\tau, z)$ is a Jacobi form of weight 0, index m.
Mathematicians have studied Jacobi forms and it can be shown that a Jacobi form of weight k, index m can be constructed as a product of: $E_4(\tau)$,$E_6(\tau)$, $\phi_{-2,1}(z,\tau)$, $\phi_{0,1}(z,\tau)$ and $\phi_{-1,2}(z,\tau)$ where E_{2k} is the Eisenstein seires and $\phi_{i,j}(z,\tau)$ is a Jacobi form of wight i, index j whose explicit expression is known and can be found in the appendix. In particular E_{2k} can be treated as a Jacobi form of weight 2k, index 0. Suppose we consider $E_4(\tau)\cdot\phi_{-2,1}(z,\tau)$, the result will be a Jacobi form of weight 4+(-2)=2, index 0+1=1.
We can apply this result to find the elliptic genus when the target space is K3. The complex dimension of K3 is 2, this implies that $Z_{K3}(\tau, z)$ should be a Jacobi form of weight 0,index 1. But the only Jacobi form of weight 0, index 1 is $\phi_{0,1}(z,\tau)$, thus we know that:

$$Z_{K3}(\tau, z) = const\cdot\phi_{0,1}(z,\tau) \tag{10}$$

When z=0, $Z_{K3}(\tau, z=0)$ coincides with the Witten index which just counts the Euler number of K_3. This fixes the constant to be 2. So the conclusion is :

$$Z_{K3}(\tau, z) = 2\cdot\phi_{0,1}(z,\tau) \tag{11}$$

CONCLUSION

This demonstrates the power of modularity in gaining extremely non-trivial information about the spectrum of N(2,2) SCFT. Similar methods can also be used to count microscopic BPS states at weak coupling and compute entropy of black hole with the same charges at strong coupling.

REFERENCE

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APPENDIX

The explicit expression for the Jacobi forms mentioned in previous section:

$$\begin{aligned} \phi_{-2,1}(z,\tau) &= \frac{\theta_1^2(z,\tau)}{\eta^6(\tau)} \\ \phi_{0,1}(z,\tau) &= 4\left(\frac{\theta_2^2(z,\tau)}{\theta_2^2(z=0,\tau)} + \frac{\theta_3^2(z,\tau)}{\theta_3^2(z=0,\tau)}\right. \\ &\quad \left. + \frac{\theta_4^2(z,\tau)}{\theta_4^2(z=0,\tau)}\right) \\ \phi_{-1,2}(z,\tau) &= \frac{\theta_1(2z,\tau)}{\eta^3(\tau)} \\ \theta_1(z,\tau) &= \theta(z+\frac{1}{2}+\frac{\tau}{2},\tau)ie^{\frac{\pi i\tau}{4}+\pi i z} \\ \theta_2(z,\tau) &= \theta(z+\frac{\tau}{2},\tau)e^{\frac{\pi i\tau}{4}+\pi i z} \\ \theta_3(z,\tau) &= \theta(z,\tau) \\ \theta_4(z,\tau) &= \theta(z+\frac{1}{2},\tau) \\ \theta(z,\tau) &= \sum_{n\in Z} e^{\pi i\tau n^2}e^{2\pi i n z} \\ \eta(\tau) &= e^{\frac{2\pi i\tau}{24}}\Pi_{n=1}^{\infty}(1-e^{2\pi i n\tau}) \end{aligned} \tag{12}$$