

## Physics 215B QFT Winter 2022 Assignment 1 – Solutions

Due 11:59pm Monday, January 10, 2022

- Please hand in your homework electronically via Canvas. The preferred option is to typeset your homework. It is easy to do and you need to learn to do it anyway as a practicing scientist. A LaTeX template file with some relevant examples is provided [here](#). If you need help getting set up or have any other questions please email me. I am happy to give TeX advice.
- To hand in your homework, please submit a pdf file through the course's Canvas website, under the assignment labelled hw 01.

Thanks in advance for following these guidelines. Please ask me by email if you have any trouble.

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1. **Brain-warmer.** Convince yourself that

$$(\partial_g)^n e^{-1/g}|_{g=0} = 0 \quad \forall n.$$

This means that a function of the form  $e^{-1/g}$  does not have a useful series expansion about  $g = 0$ .

2. **Scale invariant quantum mechanics.**

Consider the action for one quantum variable  $r$  with  $r > 0$  and

$$S[r] = \int dt \left( \frac{1}{2} m \dot{r}^2 - V(r) \right), \quad V(r) = \frac{\lambda}{r^2}.$$

- (a) Show that the (non-relativistic) mass parameter  $m$  can be eliminated by a multiplicative redefinition of the field  $r$  or of the time  $t$ . As a result, convince yourself that the physics of interest here should only depend on the combination  $m\lambda$ . Show that the coupling  $m\lambda$  is dimensionless:  $[m\lambda] = 0$ .

As before in 0+1 dimensions, demanding that  $[S_{\text{kinetic}}] = 0 = -1 + 2 - 2[\sqrt{m}r]$  implies  $[\sqrt{m}r] = -\frac{1}{2}$ . So  $0 = [\int dt \frac{\lambda}{r^2}] = -1 + [\lambda m] + 2[\sqrt{m}r] = [\lambda m]$  – we see that  $[\lambda m] = 0$ . Notice that the mass can be scaled away by redefining the variable  $r$ . The physics will only depend on  $\lambda m$ .

(b) Show that this action is *scale invariant*, i.e. show that the transformation

$$r(t) \rightarrow s^\alpha \cdot r(st) \quad (1)$$

(for some  $\alpha$  which you must determine), (with  $s \in \mathbb{R}^+$ ) is a symmetry. Find the associated Noether charge  $\mathcal{D}$ . For this last step, it will be useful to note that the infinitesimal version of (1) is ( $s = e^a$ ,  $a \ll 1$ )

$$\delta r(t) = a \left( \alpha + t \frac{d}{dt} \right) r(t).$$

With the finite form of the transformation it is easier to check that the action is invariant. Especially with a symmetry that acts on spacetime, we must be careful about active and passive issues – this problem is actually a bit of a nightmare of signs because of that. Then

$$S[s^\alpha \cdot r(st)] = \int dt \left( \frac{1}{2} m (s^\alpha \partial_t r(st))^2 - \frac{\lambda}{(s^\alpha r(st))^2} \right) \quad (2)$$

$$\stackrel{t_s \equiv st}{=} \int dt_s s^{-1} \left( \frac{1}{2} m s^2 s^{2\alpha} (\partial_{t_s} r(t_s))^2 - \lambda s^{-2\alpha} r^{-2}(s t_s) \right). \quad (3)$$

I emphasize that it is the dynamical field  $r(t)$  that transforms, not the time coordinate, which is just a dummy variable. The RHS of (3) equals  $S[r(t)]$  if  $2\alpha + 2 - 1 = 0$  and  $-2\alpha - 1 = 0$  which both require  $\alpha = -\frac{1}{2}$ . Note that this agrees with our naive dimensional analysis.

In field theory the way to find the Noether current is the following. If we know that under a transformation  $\phi \rightarrow \phi_\epsilon$  with parameter  $\epsilon$  constant in spacetime, the action does not change:  $S[\phi] = S[\phi_\epsilon]$  then if we allow  $\epsilon = \epsilon(x)$  (infinitesimal) then the variation must be proportional to derivatives of  $\epsilon$ :

$$\delta S \equiv S[\phi_{\epsilon(x)}] - S[\phi] = \int d^D x \partial_\mu \epsilon j^\mu(x)$$

for some functional of the fields  $j^\mu$ . The RHS is  $\delta S = - \int d^D x \epsilon \partial_\mu j^\mu$  by integration by parts (we assume no boundary); but this must vanish for constant epsilon! Therefore  $\partial_\mu j^\mu = 0$ .  $j^\mu$  is the conserved noether current. Its time component, integrated over space, is time independent:

$$\frac{d}{dt} Q \equiv \frac{d}{dt} \int d^{D-1} \vec{x} j^0 = - \int \nabla \cdot j = 0$$

(again we ignore boundary terms, in space). This method is superior to formulae you remember from classical mechanics (like  $Q = \frac{\partial L}{\partial \dot{r}} \delta r + L \delta t$ )

because it makes no assumptions about the dependence of the Lagrangian on  $\dot{q}$  and it doesn't require remembering anything.

In this example there is no space and we have  $j^0 = Q$ . The infinitesimal variation of  $r$  is  $\delta r(t) = a(-\frac{1}{2} + t\partial_t)r(t)$ . Under the transformation (1) with constant  $s$ , the lagrangian ( $L$  in  $S = \int dtL$ ) rescales by  $L(t) \rightarrow sL(st)$ ; this is necessary to cancel the variation of the measure  $\int dt \rightarrow \int d(st)$ . The infinitesimal statement is

$$\delta L(t) = a(1 + t\partial_t)L(t) = \frac{d}{dt}(atL).$$

But under a variation with  $a = a(t)$ , we would acquire an extra term from the variation of the kinetic term:

$$\delta\left(\frac{1}{2}m\dot{r}^2\right) = m\dot{r}\delta\dot{r} = m\dot{r}\dot{a}\left(-\frac{1}{2} + t\partial_t\right)r + \text{terms that are there when } \dot{a} = 0.$$

So the variation of  $L$  is

$$\delta L = \dots + \dot{a}\left(tL - \frac{1}{2}m\dot{r}r + mtr^2\right)$$

where again the ... is terms that would be there if  $\dot{a} = 0$ . So by the method described above we have:

$$\mathcal{D} = tL - \frac{1}{2}m\dot{r}r + mtr^2 = t\left(\frac{1}{2}m\dot{r}^2 + \frac{\lambda}{r^2}\right) - \frac{1}{2}m\dot{r}r.$$

We conclude that the Noether charge is

$$\mathcal{D} = +Ht - \frac{1}{2}rm\dot{r} = Ht - \frac{1}{2}rp.$$

We can check that on a solution of the EoM  $\dot{p} = -\partial_r H, \dot{r} = +\partial_p H$ :

$$\frac{d}{dt}\mathcal{D} = H - \frac{1}{2}r\dot{p} - \frac{1}{2}p\dot{r} = +H - \frac{1}{2}p\partial_p H + \frac{1}{2}r\partial_r H = +H - \frac{1}{2}\frac{p^2}{m} - \frac{1}{2}\frac{2\lambda}{r^2} = 0.$$

- (c) Find the position-space Hamiltonian  $\mathbf{H}$  governing the dynamics of  $r$ . Show that the Schrödinger equation is Bessel's equation

$$\left(-\frac{\partial_r^2}{2m} + \frac{\lambda}{r^2}\right)\psi_E(r) = E\psi_E(r).$$

Show that the Noether charge associated  $\mathcal{D}$  with scale transformations ( $\equiv$  dilatations) satisfies:  $[\mathcal{D}, \mathbf{H}] = -i\mathbf{H}$ . This equation says that the Hamiltonian has a definite scaling dimension, *i.e.* that its scale transformation is

$\delta\mathbf{H} = \mathbf{i}a[\mathcal{D}, \mathbf{H}] = +a\mathbf{H}$ . Note that you should not need to use arcane facts about Bessel functions, only the asymptotic analysis of the equation, in subsequent parts of the problem.

The Hamiltonian is  $\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \frac{\lambda}{r^2}$ . The quantum dilatation operator is then  $\mathcal{D} = -\frac{1}{2}\mathbf{r}\mathbf{p} + t\mathbf{H}$  (plus a possible constant term related to ordering issues of  $\mathbf{x}$  and  $\mathbf{p}$  which we can neglect). We can check that this generates the correct variation of  $\mathbf{r}$  by commutators:

$$\delta\mathbf{r} = -\mathbf{i}a[\mathcal{D}, \mathbf{r}] = \mathbf{i}a[-\frac{1}{2}\mathbf{r}\mathbf{p} + t\mathbf{H}, \mathbf{r}] = \mathbf{i}a\left(-\frac{1}{2}\mathbf{r}(-\mathbf{i}) + t\mathbf{i}\partial_t\mathbf{r}\right) = a\left(-\frac{1}{2}\mathbf{r} + t\dot{\mathbf{r}}\right) \quad \checkmark$$

(where we used the CCR  $[\mathbf{p}, \mathbf{r}] = -\mathbf{i}$ ). So

$$[\mathcal{D}, \mathbf{H}] = [-\frac{1}{2}\mathbf{r}\mathbf{p} + t\mathbf{H}, \mathbf{H}] = -\frac{1}{2}(\mathbf{r}[\mathbf{p}, \mathbf{H}] + [\mathbf{r}, \mathbf{H}]\mathbf{p}) = -\frac{1}{2}\left(+2\mathbf{i}\frac{\lambda}{r^2} + 2\mathbf{i}\frac{\mathbf{p}^2}{2m}\right) = -\mathbf{i}\mathbf{H}.$$

This is the statement that the hamiltonian has definite *scale dimension* (namely one). We found a conserved current, but it doesn't commute with the Hamiltonian. What gives? The thing that's true is that the charge is time independent – the total time derivative vanishes. But the total time derivative has two parts (using the Heisenberg equations of motion):

$$\frac{d}{dt}\mathcal{D} = \partial_t\mathcal{D} + \mathbf{i}[H, \mathcal{D}] = H - H = 0.$$

- (d) Describe the behavior of the solutions to this equation as  $r \rightarrow 0$ . [Hint: in this limit you can ignore the RHS. Make a power-law ansatz:  $\psi(r) \sim r^\Delta$  and find  $\Delta$ .]

Plugging in  $\psi(r) \sim r^\Delta$  gives

$$\Delta(\Delta - 1) + 2m\lambda = 0 \quad \implies \quad \Delta_\pm = \frac{1}{2} \pm \sqrt{2m\lambda + \frac{1}{4}}.$$

This is the leading term in an expansion in  $r$ . That is, the exact solution has a series expansion of the form

$$\psi(r) = r^{\Delta_+} \sum_{n=0}^{\infty} a_n^+ r^n + r^{\Delta_-} \sum_{n=0}^{\infty} a_n^- r^n$$

where all the coefficients  $a_n^\pm$  for  $n > 1$  are determined by  $a_1^\pm$  by the differential equation. If you plug in the ansatz  $r^{\Delta_\pm} \sum_{n=0}^{\infty} a_n^\pm r^n$  into the Schrödinger equation, you'll find that the  $a_n$ s satisfy a (two-term) recursion relation which specifies a Bessel function. This is called the Method of Frobenius for studying differential equations with a regular singular point.

- (e) What happens if  $2m\lambda < -\frac{1}{4}$ ? It looks like there is a continuum of negative-energy solutions (boundstates). This is another example of a *too-attractive* potential.

When  $2m\lambda$  passes through  $-\frac{1}{4}$  from above the two roots  $\Delta_{\pm}$  collide and move off into the complex plane. This means that the eigenfunctions *oscillate* near the origin, like  $e^{ia \log(r)}$  for some constant  $a$ , for any  $E$ . This is innocuous for  $E > 0$  where there is already a continuum of scattering states. (Recall that the behavior at large  $r$  satisfies  $-\partial_r^2 \psi = E\psi$ , so

$$\psi(r) \underset{r \rightarrow \infty}{\sim} \begin{cases} e^{i\sqrt{E}r}, & E > 0 \\ e^{-\sqrt{|E|r}}, & E < 0 \end{cases}$$

- ) But for  $E < 0$ , a good Schrodinger equation will specify a discrete set of energies at which we can integrate the wavefunction in from  $r = \infty$  without encountering a singularity. This one (at  $2m\lambda < -\frac{1}{4}$ ) instead allows any negative  $E$ . This suggests a problem.
- (f) A hermitian operator has orthogonal eigenvectors. We will show next that to make  $\mathbf{H}$  hermitian when  $2m\lambda < -\frac{1}{4}$ , we must impose a constraint on the wavefunctions:

$$(\psi_E^* \partial_r \psi_E - \psi_E \partial_r \psi_E^*)|_{r=0} = 0. \quad (4)$$

There are two useful perspectives on this condition: one is that the LHS is the probability current passing through the point  $r = 0$ .

The other perspective is the following. Consider two eigenfunctions:

$$\mathbf{H}\psi_E = E\psi_E, \quad \mathbf{H}\psi_{E'} = E'\psi_{E'}.$$

Multiply the first equation by  $\psi_{E'}^*$  and integrate; multiply the second by  $\psi_E^*$  and integrate; take the difference (maybe take the complex conjugate of the second term). Show that the result is a boundary term which must vanish when  $E = E'$ .

The boundary term is (the Wronskian):

$$\psi_E^* \partial_r \psi_E - \psi_E \partial_r \psi_E^*.$$

This is ( $i$  times) the probability current through the point  $r$ . For  $r \rightarrow 0$  the probability has nowhere to go, (and more generally for static solutions the probability should not be moving) so if this doesn't vanish there is trouble. We conclude from this analysis that when eigenstates of  $\mathbf{H}$  go like  $\psi(r) \sim r^{\Delta}$  with complex  $\Delta$ , that  $\mathbf{H}$  must not be Hermitian: probability is leaking into the hole in the potential at  $r = 0$ ! It does not describe a closed system.

(g) Show that the condition (4) is empty for  $2m\lambda > -\frac{1}{4}$ . We must impose a further condition on the eigenfunctions for  $2m\lambda < -\frac{1}{4}$  in order to satisfy (4). [It will help to impose a boundary condition at  $r = \epsilon$  for some UV cutoff  $\epsilon$ . Notice that this regulator explicitly breaks scale invariance.] Show that the resulting spectrum of boundstates has a *discrete* scale invariance.

[Cultural remark: For some reason I don't know, restricting the Hilbert space in this way is called a *self-adjoint extension*.]

This model has been studied extensively, beginning, I think, with K.M. Case, *Phys Rev* **80** (1950) 797. More recent literature includes Hammer and Swingle, [arXiv:quant-ph/0503074](https://arxiv.org/abs/quant-ph/0503074), *Annals Phys.* 321 (2006) 306-317. The associated Schrödinger equation also arises as the scalar wave equation for a field in anti de Sitter space. A recent paper which discusses connections with the renormalization group in more detail is [this one](#), by S. Paik.

If we plug in  $\psi = A_+ r^{\frac{1}{2}+i\nu} + A_- r^{\frac{1}{2}-i\nu}$  into (4) it becomes

$$0 = \nu (|A_+|^2 - |A_-|^2).$$

An example of a condition which implies (4) is simply to impose that

$$a = \psi(r)|_{r=\epsilon}$$

for some constant  $a$  and  $\epsilon \ll 1$  some *UV cutoff* on the potential. Any other choice of boundary condition at  $r = \epsilon$  will have the same qualitative effect. This is a *restriction* on the Hilbert space on which the Hamiltonian above is Hermitian. It is extra short-distance information (near the origin) about the potential, which comes with a scale:  $\epsilon$ . The UV cutoff breaks scale invariance. The original scale invariant theory was not well defined (as a closed quantum system).

The result of imposing this boundary condition is the following. We need one more piece of information about how the energy enters into the wavefunctions. We could find this by solving the equation exactly (*e.g.* plug it into Mathematica). The solution is some Bessel  $K$  ( $K$  is the one that goes to zero at large argument). More instructive is to use dimensional analysis to notice that  $E$  scales like  $r^{-2}$ ; therefore, up to some overall factor, the wavefunction must be a function of  $\sqrt{E}r$ . So we want to impose :

$$a = \psi(r)|_{r=\epsilon} = A_+ \left(\sqrt{E}r\right)^{\frac{1}{2}+i\delta} + A_- \left(\sqrt{E}r\right)^{\frac{1}{2}-i\delta}$$

where  $A_{\pm}$  are the two (real) integration constants and  $\delta \equiv \sqrt{|2m\lambda - \frac{1}{4}|}$ . A more convenient description is in terms of amplitude and phase

$$a = A_0 \epsilon^{\frac{1}{2}} \cos \left( \delta \log \epsilon \sqrt{E} + \varphi \right)$$

which is satisfied for  $E_n$  such that

$$\frac{\delta}{2} \log E \epsilon^2 = -2\pi n + \varphi$$

Or

$$E_n \propto \epsilon^{-2} e^{-\frac{4\pi n}{\delta}}.$$

The energies of the allowed boundstates form a geometric series. Again we see dimensional transmutation: the energy scale that determines the bound-state spectrum comes from the UV cutoff  $\epsilon$  we were forced to introduce.

- (h) [Extra credit] Consider instead a particle moving in  $\mathbb{R}^d$  with a central  $1/r^2$  potential,  $r^2 \equiv \vec{x} \cdot \vec{x}$ ,

$$S[\vec{x}] = \int dt \left( \frac{1}{2} m \dot{\vec{x}} \cdot \dot{\vec{x}} - \frac{\lambda}{r^2} \right).$$

Show that the same analysis applies (*e.g.* to the s-wave states) with minor modifications.

[A useful intermediate result is the following representation of (minus) the laplacian in  $\mathbb{R}^d$ :

$$\vec{p}^2 = -\frac{1}{r^{d-1}} \partial_r (r^{d-1} \partial_r) + \frac{\hat{L}^2}{r^2}, \quad \hat{L}^2 \equiv \frac{1}{2} \hat{L}_{ij} \hat{L}_{ij}, \quad L_{ij} = -\mathbf{i} (x_i \partial_j - x_j \partial_i),$$

where  $r^2 \equiv x^i x^i$ . By ‘s-wave states’ I mean those annihilated by  $\hat{L}^2$ .]

The only differences, once we go to polar coordinates, are that there are angular variables, and there is a centripetal term in the laplacian,  $\frac{\ell(\ell+1)}{r^2}$ . For s-waves,  $\ell = 0$ , we can ignore the latter complication (more generally, we can just shift  $\lambda$  by  $\frac{\ell(\ell+1)}{2m}$ ). The wave operator is then  $\nabla^2 = -r^{1-d} \partial_r (r^{d-1} \partial_r)$ . The only change is in the relation between the power-law behavior near  $r = 0$  and  $\lambda$  from studying the  $r \rightarrow 0$  asymptotics of the schrödinger equation:

$$\left( -E - \frac{1}{2m} \nabla^2 + \frac{\lambda}{r^2} \right) \psi \sim \left( -\frac{1}{2m} r^{1-d} \partial_r (r^{d-1} \partial_r) + \frac{\lambda}{r^2} + \mathcal{O}(r^2) \right) r^\Delta$$

which gives  $0 = -\Delta(\Delta + d - 2) + 2m\lambda$  and hence

$$\Delta_{\pm} = \frac{2 - d \pm \sqrt{(d-2)^2 + 8\lambda}}{2}.$$