University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2022
Assignment 2 – Solutions

Due 11:59pm Monday, January 17, 2022

Thanks in advance for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Brain-warmer.**

   Use the Clifford algebra to show that

   $$\gamma^\mu \left(x\not{p} + m\right) \gamma_\mu = -2x\not{p} + 4m$$

   where as usual $\not{p} \equiv p^\mu \gamma_\mu$. This identity will be useful in the numerator of the electron self-energy.

2. **An example of renormalization in classical physics.**

   Consider a classical scalar field in $D+2$ spacetime dimensions coupled to an *impurity* (or defect or brane) in D dimensions, located at $X = (x^\mu, 0, 0)$. Suppose the field has a self-interaction which is localized on the defect. For definiteness and calculability, we'll consider the simple (quadratic) action

   $$S[\phi] = \int d^{D+2}X \left(\frac{1}{2}\partial_M \phi(X) \partial^M \phi(X) + g\delta^2(\vec{x}_\perp)\phi^2(X)\right).$$

   Here $X^M = (x^\mu, x^i_\perp)$, $\mu = 0..D-1$, $i = 1, 2$, *i.e.* $x_\perp$ are coordinates transverse to the impurity.

   This example is from this paper by Goldberger and Wise.

   (a) What is the mass dimension of the coupling g? This is why I picked a codimension[1]-two defect.

   (b) Find the equation of motion for ϕ. Where have you seen an equation like this before?

   It's the Schrödinger equation for a particle in a 2d delta function potential.

[1] An object whose position requires specification of p coordinates has codimension p.

(c) We will study the propagator for the field in a mixed representation:

$$G_k(x,y) \equiv \langle \phi(k,x)\phi(-k,y)\rangle = \int d^D z\ e^{\mathbf{i}k_\mu z^\mu} \langle \phi(z,x)\phi(0,y)\rangle$$

– *i.e.* we go to momentum space in the directions in which translation symmetry is preserved by the defect. Find and evaluate the diagrams contributing to $G_k(x,y)$ in terms of the free propagator $D_k(x,y) \equiv \langle \phi(k,x)\phi(-k,y)\rangle_{g=0}$. (We will not need the full form of $D_k(x,y)$.) Sum the series.

I found it convenient to do this problem in Euclidean spacetime, so G and D are Euclidean propagators.

The euclidean path integral is of the form $\int D\phi e^{-S_0}e^{-V}$ where S_0 is the kinetic term and $V = \int d^{D+2}x\delta^2(x_\perp)g\phi^2$. So if we work in euclidean time, the two-point vertex is $-g\delta^{(2)(x)}$, and no $\mathbf{i}$s will appear. From the sum of diagrams of the form (just as if we had done perturbation theory in the mass)

— + —x— + —x—x— + —x—x—x— ...

we find a geometric series

$$\begin{aligned} G_k(x,y) &= D_k(x,y) - g\int d^2z_1 D_k(x,z_1)\delta^{(2)}(z_1)D_k(z_1,y) \\ &\quad + (-g)^2\int d^2z_1\int d^2z_2 D_k(x,z_1)\delta^{(2)}(z_1)D_k(z_1,z_2)\delta^{(2)}(z_2)D_k(z_2,y) + \cdots \\ &= D_k(x,y) - gD_k(x,0)D_k(0,y) + (-g)^2 D_k(x,0)D_k(0,0)D_k(0,y) \\ &\quad + (-g)^3 D_k(x,0)D_k(0,0)^2 D_k(0,y) + \cdots \\ &= D_k(x,y) - gD_k(x,0)\left(1 - gD_k(0,0) + (-g)^2 D_k(0,0)^2 + \cdots\right) D_k(0,y) \\ &= D_k(x,y) - \frac{g}{1+gD_k(0,0)} D_k(x,0)D_k(0,y). \end{aligned}$$

(d) You should find that your answer to part 2c depends on $D_k(0,0)$, which is divergent. This divergence arises from the fact that we are treating the defect as infinitely thin, as a pointlike object – the δ^2-function in the interaction involves arbitrarily short wavelengths. In general, as usual, we must really be agnostic about the short-distance structure of things. To reflect this, we introduce a regulator. For example, we can replace the fourier representation of $D_k(0,0)$ with the cutoff version

$$D_k(0,0;\Lambda) = \int_0^\Lambda đ^2 q \frac{e^{\mathbf{i}q\cdot 0}}{k^2+q^2}. \tag{1}$$

Do the integral.

Note that the formula (1) would need an extra factor of $\mathbf{i}$ if we were working in real time (in which case the interaction vertex would be $-\mathbf{i}g\delta^2(x)$, and the $\mathbf{i}$s would eat each other).

$$D_k(0,0;\Lambda) = \int_0^\Lambda đ^2q \frac{e^{\mathbf{i}q\cdot 0}}{k^2+q^2} = \frac{1}{4\pi}\log\frac{\Lambda^2+k^2}{k^2} \overset{\Lambda \gg k}{=} \frac{1}{4\pi}\log\frac{\Lambda^2}{k^2}.$$

These dimensions we're integrating here are spacelike, so there's no need for any Wick rotation.

(e) Now we renormalize. We will let the *bare coupling* g (the one which appears in the Lagrangian, and in the series from part 2c) depend on the cutoff $g = g(\Lambda)$. We wish to eliminate $g(\Lambda)$ in our expressions in favor of some measurable quantity. To do this, we impose a renormalization condition: choose some reference scale μ, and demand that[2]

$$G_\mu(x,y) \overset{!}{=} D_\mu(x,y) - g(\mu)D_\mu(x,0)D_\mu(0,y). \tag{2}$$

This equation defines $g(\mu)$, which we regard as a physical quantity. Show that (2) is satisfied if we let the bare coupling be $g(\Lambda) = g(\mu)Z$, with

$$Z = \frac{1}{1 - \frac{g(\mu)}{4\pi}\ln\left(\frac{\Lambda^2}{\mu^2}\right)}.$$

(f) Find the beta function for g,

$$\beta_g(g) \equiv \mu\frac{dg(\mu)}{d\mu},$$

and solve the resulting RG equation for $g(\mu)$ in terms of some initial condition $g(\mu_0)$. Does the coupling get weaker or stronger in the UV?

You may be bothered that we previously defined the beta function as $\Lambda\partial_\Lambda g(\Lambda)$, in terms of the cutoff dependence. In a classically scale-invariant theory, the dependence on Λ and μ is very closely tied together, since there are no other scales in the problem.

Solving for $g(\Lambda)$ gives

$$g(\Lambda) = \frac{g(\mu)}{1 - \frac{g(\mu)}{4\pi}\log\frac{\Lambda^2}{\mu^2}}.$$

[2] Note that if we worked in real time, there would be an extra $\mathbf{i}$ in front of the second term on the RHS.

Then

$$\beta_g(g) = \frac{g(\Lambda)}{\left(1 - \frac{g(\Lambda)}{4\pi}\log\frac{\Lambda^2}{\mu^2}\right)^2}\frac{g(\Lambda)}{2\pi} = \frac{g^2(\mu)}{2\pi} = \frac{g^2}{2\pi}.$$

The solution is

$$g(\mu) = \frac{g(\mu_0)}{1 - \frac{g(\mu_0)}{2\pi}\log\frac{\mu}{\mu_0}}$$

which grows with μ. Something bad happens when the denominator vanishes:

$$1 = \frac{g(\mu_0)}{2\pi}\log\frac{\mu_\star}{\mu_0}.$$

This scale $\mu_\star$ where the coupling blows up is called a *Landau pole*.

3. **Scale invariance in QFT in $D = 0 + 0$.** [I got this problem from Frederik Denef.]

A nice realization of QFT in $0 + 0$ dimensions is the statistical mechanics of a collection of non-interacting particles. The canonical partition function for a single particle (moving in one dimension) is

$$Z = \int đP dX e^{-\beta H} \propto \sqrt{T} Z_V(T) \tag{3}$$

with $H = \frac{P^2}{2} + V(X)$ and $T = 1/\beta$. The momentum integral is Gaussian and we can just do it. The partition function of N non-interacting indistinguishable particles is then $Z^N/N!$, which just multiplies the energy $U = T^2\partial_T \log Z$ by a factor of N, so we don't miss anything by focussing on the single particle.

Let's consider the case

$$V(X) = aX^2 + bX^4 + cX^6 \tag{4}$$

and figure out the important features of the temperature dependence of the thermodynamic quantities by scaling arguments.

(a) Assuming $a \neq 0, b \neq 0, c \neq 0$, find the behavior of the thermal energy U and the heat capacity $C = \partial_T U$ in the limit $T \to 0$ and in the limit $T \to \infty$ using scaling arguments. Which parts of the potential determine the respective limiting behaviors?

First, to understand the low-temperature behavior, let $x \equiv X/\sqrt{T}$, so that

$$Z_V = \int dX e^{-V(X)/T} = T^{1/2}\int dx e^{-(ax^2 + bx^4 T + cx^6 T^2)} = T^{1/2}\underbrace{\int dx e^{-ax^2}}_{\text{some number}}\underbrace{e^{-(bx^4 T + cx^6 T^2)}}_{\overset{T\to 0}{\to} 1}. \tag{5}$$

Therefore, $Z \overset{T\to 0}{\propto} T^{1/2+1/2}$. In this case the quadratic term is most important. If $Z \propto T^\alpha$ then $U = \alpha T$ and $C = \alpha$, so here $C = 1$. To understand the high-temperature behavior, let $y \equiv X/T^{1/6}$ so that

$$Z_V = \int dX e^{-V(X)/T} = T^{1/6} \int dy e^{-(cy^6 + ax^2/T^{2/3} + bx^4/T^{2/3})} = T^{1/6} \int dy e^{-cy^6} \underbrace{e^{-(ax^2/T^{2/3} + bx^4/T^{2/3})}}_{\overset{T\to\infty}{\to} 1}. \tag{6}$$

So at high temperatures $C \to \frac{1}{2} + \frac{1}{6}$. At high temperature, the particle can explore the whole potential and the highest power in the potential is what matters.

(b) If some of the couplings a, b, c vanish, the low or high temperature scaling behavior may change. For example, what is the heat capacity at low temperature when $a = 0, b \neq 0$?

In this case, the quartic term dominates and $Z_V \sim T^{1/4}$ and $C = 3/4$.

A word about notation: the symbol $\sim$ is usually used by physicists to indicate a scaling relationship, where the constant prefactors are neglected. The relation we derive here for C however is an equality in the relevant regime of temperatures – the constant is the thing that matters.s

(c) When b is sufficiently large (and $a \neq 0, c \neq 0$), there will be an intermediate temperature regime over which the heat capacity is again constant, but different from the low- and high-temperature limits. What is this heat capacity?

Same as the previous part.

(d) In general, we can think of the change of C with T as a kind of classical renormalization group (RG) flow, interpolating between 'fixed points' where C becomes constant. In general, these fixed points correspond to potentials $V(X)$ with a scaling symmetry $V(\lambda^\Delta X) = \lambda V(X)$ for some choice of scaling dimension Δ of X. What is the heat capacity for a fixed point with scaling dimension Δ for X?

$$Z_V = \int dX e^{-V(X)/T} = \int dX e^{-V(T^{-\Delta}X)} = T^\Delta \underbrace{\int dx e^{-V(x)}}_{\text{indep of } T} \tag{7}$$

with $x \equiv T^{-\Delta}X$. So $Z \propto T^{1/2+\Delta}$ and $C = \Delta + \frac{1}{2}$.

(e) Borrowing more language of the renormalization group, we can classify deformations $\delta V(X) = \epsilon X^m$ of a fixed point $V(X) \propto X^{2n}$ as irrelevant, marginal, or relevant, depending on whether the deformation becomes dominant or negligible in the IR limit, *i.e.* in the limit of low T. Here are below

ϵ can take on any value, not necessarily small. Restricting to deformations with an $X \to -X$ symmetry, what are the relevant and irrelevant deformations of $V(X) = X^{2n}$? (Note that a deformation $\delta V = \epsilon X^{2n}$ can be absorbed into a redefinition of X, which does not change the heat capacity.)

Lower powers than $2n$ are relevant, higher powers are irrelevant.

(f) The T-dependence of correlation functions (here, expectation values of powers of X) at fixed points is also determined by the scaling properties. What is the T-dependence of $\langle X^k \rangle$ at a fixed point $V(X) = X^{2n}$?

$$\langle X^k \rangle = \frac{\int dX X^k e^{-V(X)/T}}{Z_V} = \frac{T^{\Delta(1+k)} \int dx x^k e^{-V(x)}}{T^\Delta} \propto T^{\Delta k}$$

where $\Delta = \frac{1}{2n}$ is the scaling dimension of X.

(g) Non-polynomial $V(X)$ can be considered as well. For example, what is the heat capacity at small and large T for $V(X) = (1 + X^2)^{1/n}$?

Since this function still grows at large X, the high-temperature behavior is dominated by the large-X behavior where $V(X) \sim X^{2/n}$, so $\Delta = n/2$. At low temperature, we Taylor expand in small X to find $V(X) \sim 1 + X^2/n$ and find $\Delta = 1/2$.

4. **Meson scattering.** Consider again Yukawa theory with fermions, with

$$\mathcal{L} = \bar{\Psi}\left(\mathbf{i}\partial\!\!\!/ - m\right)\Psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 + \mathcal{L}_{\text{int}}$$

and $\mathcal{L}_{\text{int}} = g\bar{\Psi}\Psi\phi$.

(a) Consider the correction to the process $\phi\phi \to \phi\phi$ coming from a fermion loop. What counterterm is required to renormalize this interaction? (You don't need to actually do the integral for this problem.)

The one-loop contributions come from a fermion loop with four ϕs attached by the Yukawa vertex. There are various such diagrams related by exchanging the external momenta. A counterterm which can absorb its cutoff dependence is a ϕ^4 term.

(b) Do you need a cutoff-dependent counterterm of the form $\delta_3\phi^3$ in this theory?

It seems like you might, since you can draw a fermion loop and attach three external legs. Naively this goes like $\int \frac{d^Dp}{p^3}$ which is divergent for $D \geq 3$. However, the spinor trace vanishes at zero external momentum. This means the leading contribution must be proportional to powers of the external momenta and therefore (by Taylor expansion) is proportional to $\int \frac{d^Dp}{p^5}$ which is finite for $D \leq 5$.

A better answer might be to argue that there is a $\phi \to -\phi$ symmetry, which would forbid a ϕ^3 interaction. Such a symmetry would have to act on ψ as well in order to preserve the Yukawa term. It would have to act by a discrete symmetry under which $\bar{\psi}\psi$ is odd. As you can remind yourself in Peskin §3, there is such an operation. But – the mass term $m\bar{\psi}\psi$ would be odd under this term. This shows that the answer must be proportional to m, and then dimensional analysis shows that it is finite.

Note that an on-shell process $\phi \to \phi\phi$ is forbidden by kinematics (i.e. energy and momentum conservation) unless ϕ is massless. This doesn't mean that we wouldn't need a counterterm for it if it were cutoff dependent: it can appear as part of a more complicated process, such as $\phi\phi \to \phi\phi$.

5. **Electron-photon scattering at low energy.** [This is an optional bonus problem for those of you who wish to experience some of the glory of tree-level QED.]

   Consider the process $e\gamma \to e\gamma$ in QED at leading order.

   (a) Draw and evaluate the two diagrams.

   (b) Find $\frac{1}{4}\sum_{\text{spins,polarizations}} |\mathcal{M}|^2$.

   (c) Construct the two-body final-state phase space measure in the limit where the photon frequency is $\omega \ll m$ (the electron mass), in the rest frame of the electron. I suggest the following kinematical variables:

   $$p_1 = (\omega, 0, 0, \omega), p_2 = (m, 0, 0, 0), p_4 = (\omega', \omega' \sin\theta, 0, \omega' \cos\theta), p_3 = p_1 + p_2 - p_4 = (E', p')$$

   for the incoming photon, incoming electron, outgoing photon and outgoing electron respectively.

   (d) Find the differential cross section $\frac{d\sigma}{d\cos\theta}$ as a function of ω, θ, m. (The expression can be prettified by using the on-shell condition $p_3^2 = m^2$ to relate ω' to ω, θ. It is named after Klein and Nishina.) Compare to experiment.

   (e) Show that the limit $E \ll m$ gives the (Thomson) scattering cross section for classical electromagnetic radiation from a free electron.