University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2022
Assignment 3 – Solutions

Due 11:59pm Monday, January 24, 2022

Thanks in advance for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Brain-warmer.**

   Prove the Gordon identities

$$\bar{u}_2 \left(q^\nu \sigma_{\mu\nu}\right) u_1 = \mathbf{i}\bar{u}_2 \left((p_1 + p_2)_\mu - (m_1 + m_2)\gamma_\mu\right) u_1$$

   and

$$\bar{u}_2 \left((p_1 + p_2)^\nu \sigma_{\mu\nu}\right) u_1 = \mathbf{i}\bar{u}_2 \left((p_2 - p_1)_\mu - (m_2 - m_1)\gamma_\mu\right) u_1$$

   where $q \equiv p_2 - p_1$ and $\not{p}_1 u_1 = m_1 u_1, \bar{u}_2 \not{p}_2 = m_2 \bar{u}_2$, using the definitions and the Clifford algebra.

2. **Pauli-Villars practice.**

   Consider a field theory of two scalar fields with

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{2}\Phi\Box\Phi - \frac{1}{2}M^2\Phi^2 - g\phi\Phi^2 + \text{counterterms}.$$

   Compute the one-loop contribution to the self-energy of Φ. Use a Pauli-Villars regulator – introduce a second copy of the ϕ field of mass Λ with the wrong-sign propagator.

$$\Sigma_\Phi(p) = \int đ^D k \frac{\mathbf{i}}{k^2 - m^2} \frac{\mathbf{i}}{(k+p)^2 - M^2} (-\mathbf{i}g)^2 \tag{1}$$

$$= g^2 \int_0^1 dx \int đ^D k \frac{1}{\left((1-x)(k^2 - m^2) + x((k+p)^2 - M^2)\right)^2} \tag{2}$$

$$= g^2 \int_0^1 dx \int \frac{đ^D \ell}{(\ell^2 - \Delta + \mathbf{i}\epsilon)^2}, \quad \ell = k - px, \Delta = xM^2 + (1-x)m^2 - p^2 x(1-x) \tag{3}$$

$$\equiv g^2 \int_0^1 dx \mathcal{J}(\Delta(m)) \tag{4}$$

The Pauli-Villars regulator replaces the ϕ propagator by

$$\frac{\mathbf{i}}{p^2 - m^2} \rightsquigarrow \frac{\mathbf{i}}{p^2 - m^2} - \frac{\mathbf{i}}{p^2 - \Lambda^2}$$

so that the self energy is replaced by

$$\begin{aligned}
\Sigma_\Phi(p) &= g^2 \int_0^1 dx \left(\mathcal{J}(\Delta(m)) - \mathcal{J}(\Delta(\Lambda))\right) && (5)\\
&= \frac{g^2}{8\pi^2}\int_0^1 dx \log \frac{\Delta(\Lambda^2)}{\Delta(m^2)} && (6)\\
&= \frac{g^2}{8\pi^2}\int_0^1 dx \log \frac{\Delta(\Lambda^2)}{\Delta(m^2)} && (7)\\
&\overset{\Lambda \gg \text{everyone}}{=} \frac{g^2}{8\pi^2}\int_0^1 dx \log \left(\frac{x\Lambda^2}{xM^2 + (1-x)m^2 - p^2 x(1-x)}\right) && (8)
\end{aligned}$$

Actually there is also second diagram, where the fermion emits a single scalar which ends at a fermion bubble:

$$-\mathbf{i}\Sigma_\Phi^{\text{tadpole}}(p) = (-\mathbf{i}g)^2 \int đ^4 k \frac{\mathbf{i}}{k^2 - M^2}\frac{\mathbf{i}}{-m^2}.$$

This is independent of the external momentum, and so only contributes to the mass renormalization. A complication that arises here is that the loop contains only a fermion propagator, so our PV regulator involving only a heavy scalar above will not regularize this divergence. We must also add a heavy fermion ghost field. (Such a step is also required to regulate the corrections to the scalar propagator from a fermion bubble.) I'm going to ignore this diagram below.

Determine the counterterms required to impose that the Φ propagator has a pole at $p^2 = M^2$ with residue 1.

To do this, expand (8) about $p^2 = M^2$:

$$\Sigma_\Phi(p) = \frac{g^2}{8\pi^2}\int_0^1 dx \log \frac{x\Lambda^2}{M^2(1-x)^2 + m^2 x} + (p^2 - M^2)\frac{g^2}{8\pi^2}\int_0^1 dx \frac{x(1-x)}{M^2(1-x)^2 + m^2 x} \quad (9)$$

$$\equiv S_1 + (p^2 - M^2) S_2 \quad (10)$$

and do the x integrals. The mass correction depends on the cutoff like $\log\Lambda$, but δ_Z is independent of the cutoff. The actual expressions (which Mathematica can tell you if you are patient enough) are not very illuminating and I don't want to type them, but it's worth noticing that they are singular when $m = 2M$. Why? When $m = 2M$ the intermediate state can be on shell.

The total contribution to Σ, including the counterterms $\mathcal{L}_{ct} = -\delta Z \frac{1}{2}(\partial\phi)^2 + \delta_{M^2}\frac{1}{2}\Phi^2$ (note my silly sign convention) is

$$0 + (p^2 - M^2)0 + \mathcal{O}(p^2 - M^2)^2 \stackrel{!}{=} \Sigma(p) - \delta_{M^2} - p^2\delta_Z \tag{11}$$

$$= S_1 + (p^2 - M^2)S_2 - \delta_{M^2} - p^2\delta_Z + \mathcal{O}(p^2 - M^2)^2 \tag{12}$$

$$= S_1 - M^2 S_2 - \delta_{M^2} + p^2(S_2 - \delta_Z) + \mathcal{O}(p^2 - M^2)^2. \tag{13}$$

We conclude that we need to set

$$\delta_M^2 = S_1 - M^2 S_2, \quad \delta_Z = S_2$$

to satisfy the stated renormalization conditions. Notice that in this process, we not only remove the cutoff dependence, but we also determine the *finite* parts of the counterterms.

3. **Bosons have worse UV behavior than fermions.**

Consider the Yukawa theory

$$S[\phi,\psi] = -\int d^Dx \left(\frac{1}{2}\phi\left(\Box + m_\phi\right)\phi + \bar{\psi}\left(-\partial\!\!\!/ + m_\psi\right)\psi + y\phi\bar{\psi}\psi + \frac{g}{4!}\phi^4\right) + \text{counterterms}.$$

(a) Show that the superficial degree of divergence for a diagram $\mathcal{A}$ with B_E external scalars and F_E external fermions is

$$D_{\mathcal{A}} = D + (D-4)\left(V_g + \frac{1}{2}V_y\right) + B_E\left(\frac{2-D}{2}\right) + F_E\left(\frac{1-D}{2}\right) \tag{14}$$

where V_g and v_y are the number of ϕ^4 and $\phi\bar{\psi}\psi$ vertices respectively.

All the discussion below is about one loop diagrams.

(b) Draw the diagrams contributing to the self energy of both the scalar and the spinor in the Yukawa theory.

(c) Find the superficial degree of divergence for the scalar self-energy amplitude and the spinor self-energy amplitude.

(d) In the case of $D = 3+1$ spacetime dimensions, show that (with a cutoff on the Euclidean momenta) the spinor self-energy is actually only logarithmically divergent. (This type of thing is one reason for the adjective 'superficial'.)

Hint: the amplitude can be parametrized as follows: if the external momentum is p^μ, it is

$$\mathcal{M}(p) = A(p^2)\not{p} + B(p^2).$$

Show that $B(p^2)$ vanishes when $m_\psi = 0$.

See Zee, page 180. In four dimensions, the scalar self-energy has $\mathcal{D} = 2$ and indeed depends quadratically on the cutoff. The fermion self-energy has $\mathcal{D} = 1$, but the would-be leading divergence of the fermion self-energy is an integral with an odd integrand and therefore vanishes, leaving behind a mere log.

A better argument for this conclusion follows from the *chiral transformation* $\Psi \to e^{\mathbf{i}\gamma^5\alpha}\Psi$, which becomes a symmetry when the fermion mass is zero. This means that the correction to the fermion mass $B(0)$ must be proportional to the mass itself (it must go to zero when the mass goes to zero, and must be analytic in the mass for some reason I am unable to summon at the moment). Combining this statement with the dimensional analysis above, we conclude that there cannot be linear dependence on the cutoff.

4. **Dimension-dependence of dimensions of couplings.**

   (a) In what number of space dimensions does a four-fermion interaction such as $G\bar{\psi}\psi\bar{\psi}\psi$ have a chance to be renormalizable? Assume Lorentz invariance.

   [optional] Generalize the formula (14) for $D_\mathcal{A}$ to include a number V_G of four-fermion vertices.

   I find

   $$D_\mathcal{A} = D + (D-4)\left(V_g + \frac{1}{2}V_y\right) + (D-2)V_G + B_E\left(\frac{2-D}{2}\right) + F_E\left(\frac{1-D}{2}\right)$$

   Therefore the four-fermion interaction is scale invariant in $D = 2$ spacetime dimensions.

   (b) If we violate Lorentz invariance the story changes. Consider a non-relativistic theory with kinetic terms of the form $\int dt d^d x \left(\psi^\dagger \left(\mathbf{i}\partial_t - D\nabla^2\right)\psi\right)$. (Here D is a dimensionful constant. In a relativistic theory we relate dimensions of time and space by setting the speed of light to one; here, there is no such thing, and we can choose units to set D to one.) For what number of space dimensions might the four-fermion coupling be renormalizable?

   You actually already know the answer to this from our study in the first lecture of the delta function potential. The $\bar{\psi}\psi\bar{\psi}\psi$ is exactly such a contact interaction between two particles. So it is marginal when $d = 2$. Alternatively, you can count inverse-length dimensions of the time-derivative term to learn that $[\psi] = d/2$, and of the $\vec{\nabla}^2$ term to learn that $[t] = -2$, i.e. we must scale t twice as fast as space to make the free theory scale invariant. Then $0 = [G\int dtd^dx\bar{\psi}\psi\bar{\psi}\psi] = [G] - 2 - d + 2d$ gives $[G] = 2 - d$. If $d > 2$ it is an irrelevant perturbation of the free theory.

(c) In the previous example, the scale transformation preserving the kinetic terms acted by $t \to \lambda^2 t, x \to \lambda x$. More generally, the relative scaling of space and time is called the *dynamical exponent* z ($z = 2$ in the previous example). Suppose that the kinetic terms are first order in time and quadratic in the fields. Ignoring difficulties of writing local quadratic spatial kinetic terms, what is the relationship between d and z which gives scale invariant quartic interactions? What if the kinetic terms are second order in time (as for scalar fields)?

To get dynamical exponent z with first-order-in-time derivatives, we'd need a kinetic term like

$$S_0 = \int dtd^dx\bar{\psi}\left(\mathbf{i}\partial_t - \nabla^z\right)\psi.$$

So $[\psi] = -d/2$ still, but $[t] = -z$. Therefore $0 = [G\int dtd^dx\bar{\psi}\psi\bar{\psi}\psi] = [G] - z - d + 2d$ gives $[G] = z - d$ and it is scale invariant if $d = z$.

With second-order-in-time derivatives, we have

$$S_0 = \int dtd^dx\phi\left(\partial_t^2 - \nabla^{2z}\right)\phi$$

so $0 = -z - d + 2z + 2[\phi]$ says $[\phi] = (d - z)/2$, and so $0 = [g\int dtd^dx\phi^4] = -z - d + 4(d - z) + [g]$ says $[g] = 3z - d$). It is classically scale invariant if $d = 3z$.

5. **Scale invariance in QFT in $D = 0 + 0$, part 2.** [I got this problem from Frederik Denef.]

The story is more interesting if there is more than one field, *i.e.* if we consider the statistical mechanics of a particle moving in more than one dimension. Consider the example of two degrees of freedom with Hamiltonian

$$H = \frac{1}{2}P_X^2 + \frac{1}{2}P_Y^2 + V(X,Y), \qquad V(X,Y) = aX^4 + bY^8 \tag{15}$$

for some nonzero constants a, b.

(a) This potential again has a scaling symmetry $V(\lambda^{1/4}X, \lambda^{1/8}Y) = \lambda V(X,Y)$. As a result, the model describes a fixed point, with constant heat capacity. Find the heat capacity.

$$Z = \int d^2 P e^{-P^2/T} \int dX dY e^{-V(X,Y)/T} \sim T T^{1/4+1/8} \int dx dy e^{-V(x,y)} \sim T^{11/8}$$

where $x \equiv T^{-1/4}X, y = T^{-1/8}Y$. We found on the previous problem set that if $Z \sim T^\alpha$ then $C_V = \alpha$, so here $C_V = 11/8$.

(b) Restricting to deformations with independent symmetries under $X \to -X$ and $Y \to -Y$, and using the basic scaling properties of the deformations under the above scaling symmetry, what are the relevant, marginal and irrelevant deformations? (Note that in this case there are true marginal deformations that cannot be absorbed into the normalization of X and Y.)

The question of whether δV is relevant is whether it changes the low-temperature physics compared to the fixed point behavior. In the position integral, we have

$$\int dX dY e^{-V_0(X,Y)/T + \delta V(X,Y)/T} = T^{3/8} \int dx dy e^{-V_0(x,y) + \delta V(T^{1/4}x, T^{1/8}y)/T}.$$

For $\delta V = X^{2m} Y^{2n}$, the extra term in the exponent is

$$\delta V(T^{1/4}x, T^{1/8}y)/T = T^{-1} T^{2m/4} x^{2m} T^{2n/8} y^{2n} = T^{\frac{2m+n-4}{4}} x^{2m} y^{2n}.$$

So the condition for this to be relevant is that the power of T is negative

$$2m + n < 4.$$

There are now relevant interactions with $2m + n = 4$; these cannot be absorbed into field redefinitions if m and n are both nonzero. (As a check note that the terms in the fixed-point potential are counted as marginal; changes in these can be absorbed by field redefinitions.)

(c) How does $\langle X^k Y^l \rangle$ depend on T at a fixed point satisfying $V(\lambda^{\Delta_X} X, \lambda^{\Delta_Y} Y)$?

$$\langle X^k Y^l \rangle = \frac{\int dX dY X^k Y^l e^{-V(X,Y)/T}}{Z_V} = \frac{T^{\Delta_X(1+k) + \Delta_Y(1+l)} \int dx dy x^k y^l e^{-V(x,y)}}{T^{\Delta_X + \Delta_Y}} \propto T^{\Delta_X k + \Delta_Y l}.$$

A generic relevant deformation of (15) will flow to a Gaussian fixed point $V(X,Y) \sim X^2 + Y^2$ in the IR. Some other, more fine-tuned deformations will flow to other fixed points. For example, $\delta V(X,Y) = \epsilon Y^4$ will flow to $V(X,Y) = X^4 + Y^4$. But something more interesting happens for $\delta V(X,Y) = \epsilon X^2 Y^2$. We'll study this more on the next homework.