University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2022
Assignment 4 – Solutions

Due 11:59pm Monday, January 31, 2022

Thanks in advance for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

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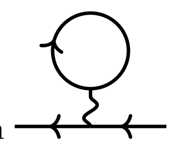
1. **Brain-warmer.** Check that $(\Delta_T)^\mu_\rho \equiv \delta^\mu_\rho - \frac{q^\mu q_\rho}{q^2}$ is a projector onto momenta transverse to q^ρ.

   This requires showing both that $\Delta q = 0$ and that $\Delta^2 = \Delta$.

2. **Tadpole diagrams.**

   (a) Why don't we worry about the following diagram  as a correction to the electron self-energy in QED?

   It has to vanish by Lorentz symmetry: the object  would be a source j^μ for the electromagnetic field in the vacuum. At one loop, we can check that $\int đ^4 k \mathrm{tr}\gamma^\mu \frac{\not{k}+m}{k^2-m^2} = 0$ by $\mathrm{tr}\gamma^\mu = 0$ and Lorentz symmetry, $\int^d 4k k^\mu f(k^2) = 0$. The one-point function for the photon also has to vanish by charge-conjugation symmetry (in fact any odd-point function of the photon does for the same reason; this is called Furry's theorem).

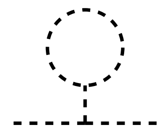
   More generally, a *tadpole diagram* – a diagram with a single field line coming out of it – represents a source for the field. When we developed our Feynman rules, we expanded around a minimum of the potential for the field, and this is why there is no one-point vertex in the Feynman rules. A tadpole diagram is saying that radiative effects are producing a shift in the minimum of the potential. The (quadratic part of the) action wants to change to $\int ((\partial A)^2 - m_\gamma^2 A^2 + Aj)$. The equations of motion for the zero-momentum field tell us that the minimum is at $A = j/m_\gamma^2$. In the case of a massless field, the shift is arbitrarily large (in this linear approximation). This is the source of the IR divergence in the tadpole diagram as $m_\gamma \to 0$. In QED, this is moot because $j = 0$.

For the remainder of the problem, we consider ϕ^3 theory with a (small) mass:

$$S = \int d^D x \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 \right).$$

(b) Notice that unlike ϕ^4 theory (or QED), there is no symmetry that forbids a one-point function for the scalar. Why don't we lose generality by not adding a term linear in ϕ to the Lagrangian?

We can shift it away by a field redefinition, $\phi \to \phi - a$. It is convenient to choose a to make the linear term vanish, since then the solution to the equations of motion has $\phi_0 = 0$.

(c) Now think about the following contribution to the scalar self-energy: 

Show that in the limit $m \to 0$ there is an IR divergence. By thinking about the significance for the scalar potential of this part of the diagram  explain the meaning of this divergence.

The object  is a one-point function for the scalar. As explained in the answer to the previous part of the problem, the presence of such a one-point function ($V_{\text{eff}} \ni v\phi$, with $v \propto g$) means we are doing perturbation theory about a configuration which is not a solution to the equations of motion at order g. The correct solution to the equations of motion is ϕ_0 with $0 = m^2\phi_0 + v$ so $\phi_0 = -v/m^2$, which diverges when $m \to 0$. This is the origin of the IR divergence – the field theory is trying to find its minimum which, when $m \to 0$, is arbitrarily far away in field space.

3. **Symmetry is attractive.** Consider a field theory in $D = 3 + 1$ with two scalar fields with the same mass which interact via the interaction

$$V = -\frac{g}{4!}\left(\phi_1^4 + \phi_2^4\right) - \frac{2\lambda}{4!}\phi_1^2\phi_2^2.$$

(a) Show that when $\lambda = g$ the model possesses an $\mathsf{O}(2)$ symmetry.

At this special point, the potential is $(\phi_1^2 + \phi_2^2)^2$, which depends only on the distance from the origin of the field space.

(b) Will you need a counterterm of the form $\phi_1\phi_2$ or $\phi_1\Box\phi_2$ (for general g, λ)? If not, why not?

A very important point: such terms can't be generated because they violate the $\mathbb{Z}_2$ symmetry which takes $(\phi_1, \phi_2) \to (-\phi_1, \phi_2)$. In general, radiative effects (*i.e.* loops) will not violate symmetries of the bare action. Exceptions to this statement are called *anomalies*; this only happens when no regulator preserves the symmetry in question.

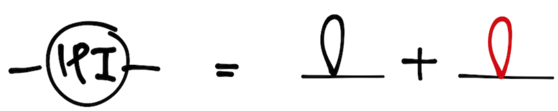
(c) Renormalize the theory to one loop order by regularizing (for example with a euclidean momentum cutoff or Pauli Villars), adding the necessary counterterms, and imposing a renormalization condition on the propagators (consider the case where the scalars are both massless) and $2 \to 2$ scattering amplitudes at some values of the kinematical variables s_0, t_0, u_0. Feel free to re-use our results from ϕ^4 theory where appropriate.

I'll use a hard euclidean momentum cutoff since then we can reuse our results from ϕ^4 theory. To save typing let me define $L(x) \equiv \frac{1}{32\pi^2} \log x$. Every loop integral we will encounter is the same as in the pure massless ϕ^4 theory that we did in lecture.

The symmetry that interchanges $\phi_1 \leftrightarrow \phi_2$ guarantees that their self-couplings g (and the masses) stay equal (using the same principle as above). This means we have only three counterterms to determine altogether: δ_{m^2} and two four-point counterterms $(\delta_g, \delta_\lambda)$. That is, we have to impose two renormalization conditions on the four-point functions.

First an annoying point: with the given normalization, the 1122 vertex is actually $-\mathbf{i}\lambda/3$.

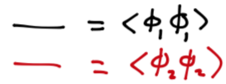
The self-energy for ϕ_1 is

$-\mathbf{i}\Sigma(p^2) =$  $+.. = -\mathbf{i}(g+\lambda/3)c\Lambda^2 + \mathcal{O}(g,\lambda)^2$

where c is a numerical constant that I can't remember right now and which we don't need. To put the pole at $p^2 = m_P^2 = 0$, we need the bare mass to be

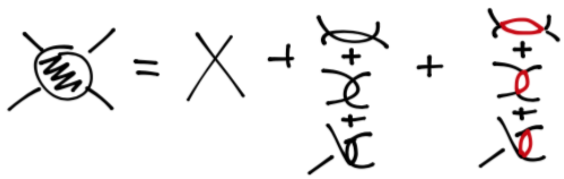
$$m^2(\Lambda) = -\Sigma(p^2 = 0) = (g + \lambda/4)c\Lambda^2.$$

As in ϕ^4 theory, there is no wavefunction renormalization at one loop because Σ is independent of p^2.

There are three different $2 \to 2$ scattering processes to consider: $11 \to 11, 11 \to 22, 12 \to 12$. (The corrections to $22 \to 22$ are the same as those for $11 \to 11$, and similiarly $22 \to 11$ is the same as $11 \to 22$, by the exchange symmetry.) Then using the notation 

we have

$$\mathcal{M}_{11\leftarrow 11} = -g + (g^2 + \left(\frac{\lambda}{3}\right)^2)(L(s/\Lambda^2) + L(t/\Lambda^2) + L(u/\Lambda^2)) + \delta_g \quad (1)$$



(2)

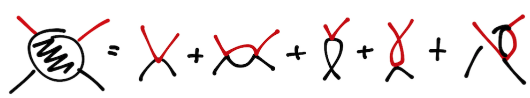
The λ^2 term involves ϕ_2 running in the loop. (Note that I am writing $\mathbf{i}\mathcal{M} = -\mathbf{i}g + (-\mathbf{i}g)^2$... and dividing the BHS by $\mathbf{i}$.) Beware the symmetry factor of $\frac{1}{2}$ in each loop diagram.

$$\mathcal{M}_{22\leftarrow 11} = -\frac{\lambda}{3} + \frac{\lambda}{3}g2L(s/\Lambda^2) + \left(\frac{\lambda}{3}\right)^2\left(2L(t/\Lambda^2) + 2L(u/\Lambda^2)\right) + \delta_\lambda \quad (3)$$

(4)

where the 2 in the s-channel term is from the fact that either ϕ_1 or ϕ_2 can run in the loop. The last two diagrams have a different symmetry factor from the others, since we can't exchange the two propagators in the loop – so they get an extra factor of 2.

$$\mathcal{M}_{12\leftarrow 12} = -\frac{\lambda}{3} + \left(\frac{\lambda}{3}\right)^2\left(2L(s/\Lambda^2) + 2L(u/\Lambda^2)\right) + 2\frac{\lambda}{3}gL(t/\Lambda^2) + \delta_\lambda \quad (5)$$



(6)

Using the renormalization conditions $\mathcal{M}_{11\leftarrow 11}(s_0 = t_0 = u_0) = -g_P$ and $\mathcal{M}_{22\leftarrow 11}(s_0 = t_0 = u_0) = -\frac{\lambda_P}{3}$ we find

$$\lambda(\Lambda) \equiv \lambda + \delta_\lambda = \lambda_P + \lambda_P 2g_P L + 4\frac{\lambda_P^2}{3}L + \mathcal{O}(\lambda_P, g_P)^2 \quad (7)$$

$$g(\Lambda) \equiv g + \delta_g = g_P + \left(g_P^2 + \left(\frac{\lambda_P}{3}\right)^2\right)3L + \mathcal{O}(\lambda_P, g_P)^2 \quad (8)$$

where $L \equiv L(s_0/\Lambda^2)$. We've solved for the couplings perturbatively, to second order in both, which means we ignored the difference between *e.g.* g and g_P in the quadratic term, as we must. From now on I will drop the P subscripts on the physical coupling.

Notice that we would get the same answer if we defined λ_P by fixing a value of $\mathcal{M}_{12\leftarrow 12}$ instead. This is because of crossing symmetry.

(d) Consider the limit of low energies, *i.e.* when $s_0, t_0, u_0 \ll \Lambda^2$ where Λ is the cutoff scale. Tune the location of the poles in both propagators to $p^2 = 0$. Show that the coupling goes to the $\mathsf{O}(2)$-symmetric value if it starts nearby (nearby means $\lambda/g < 3$).

A nice trick for doing this is to compute the beta functions.

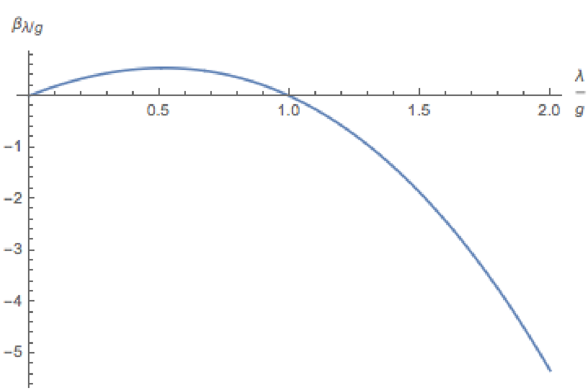
$$\beta_g \equiv 32\pi^2\Lambda^2\partial_{\Lambda^2} g(\Lambda) = 3\left(g^2 + \left(\frac{\lambda}{3}\right)^2\right), \quad \beta_\lambda \equiv 32\pi^2\Lambda^2\partial_{\Lambda^2}\lambda(\Lambda) = \left(2\lambda g + 4\frac{\lambda^2}{3}\right)$$

where I've pulled out a factor of $32\pi^2$ in the definition of β for convenience – it only affects how fast the flow happens. A useful check is that if we set $\lambda = 0$, we reproduce the beta function for ϕ^4 theory, $\beta_g = +3g^2$ (the 3 comes from the 3 different channels).

To look at the relative flow of g and λ let's compute

$$\beta_{\lambda/g} \equiv 8\pi^2\Lambda^2\partial_{\Lambda^2}\frac{\lambda}{g} = \frac{1}{g^2}\left(g\beta_\lambda - \lambda\beta_g\right) \propto \left(-\frac{\lambda^3}{3} - \frac{5}{3}g\lambda^2 + 2g^2\lambda\right) = \frac{1}{3}\lambda(\lambda - g)(\lambda + 6g).$$

This looks like this:



with the convention I'm using, positive β means that as we increase Λ, the coupling decreases. This means that the couplings approach the point $g = \lambda$ as $\Lambda \to \infty$ fixing g_P, λ_P. This is the case as long as we start with $\lambda/g < 3$.

4. **Bremsstrahlung.** Show that the number of photons per decade of wavenumber produced by the sudden acceleration of a charge is (in the relativistic limit $-q^2 \gg m^2$)

$$f_{IR}(q^2) = 2\frac{\alpha}{\pi}\ln\left(\frac{-q^2}{m^2}\right),$$

where $q_\mu = p'_\mu - p_\mu$ is the change of momentum and m is the mass of the charge.

This is explained well on pages 177-182 of Peskin. The energy comes out to

$$U = \int đ^3 k \frac{2\alpha}{\pi} \ln\left(\frac{-q^2}{m^2}\right) = 2\int đ^3 k k N_k$$

where N_k is the number density of photons of momentum k of each polarization, and the RHS used the fact that each photon of momentum k carries energy k. (The 2 comes from two polarizations for each momentum) Then the *number* of photons is

$$\mathcal{N} = \int \frac{dk}{k}\frac{\alpha}{\pi} \ln\left(\frac{-q^2}{m^2}\right) = \int d\log k \frac{\alpha}{\pi} \ln\left(\frac{-q^2}{m^2}\right)$$

and hence $2\frac{\alpha}{\pi}\ln\left(\frac{-q^2}{m^2}\right)$ is the total number of photons per decade of wavenumber. (Note that the integral over k here actually diverges; this is an artifact of the approximation that the momentum change is instantaneous.)

5. **Scale invariance in QFT in $D = 0 + 0$, part 3.** [I got this problem from Frederik Denef and it is optional but strongly encouraged.]

   We continue our study of QFT in $D = 0 + 0$ with two fields:

$$Z = \int dP_X dP_Y dX dY e^{-H/T}.$$

   Let's start by considering again

$$H = \frac{1}{2}P_X^2 + \frac{1}{2}P_Y^2 + V(X,Y), \qquad V(X,Y) = aX^4 + bY^8 \tag{9}$$

   for some nonzero constants a, b.

   A generic relevant deformation of (9) will flow to a Gaussian fixed point $V(X,Y) \sim X^2 + Y^2$ in the IR. Some other, more fine-tuned deformations will flow to other fixed points. For example, $\delta V(X,Y) = \epsilon Y^4$ will flow to $V(X,Y) = X^4 + Y^4$. But something more interesting happens for $\delta V(X,Y) = \epsilon X^2 Y^2$. This deformation is a relevant perturbation of (9) in the sense that $\delta V(\lambda^{1/4}X, \lambda^{1/8}Y) = \lambda^\kappa V(X,Y)$ with $\kappa = 3/4 < 1$. But it is not true that the model simply flows to a fixed point with $V \propto X^2Y^2$ in the IR. That's because the model with such a potential has a divergent partition function: $\int_{-\infty}^{\infty} dX \int_{-\infty}^{\infty} dY e^{-\epsilon X^2 Y^2/T} \propto \sqrt{\frac{T}{\epsilon}} \int \frac{dX}{|X|} = \infty$. We cannot throw away the higher-order terms because they regulate the large-X and large-Y behavior of the integral. Thus, in this model, the UV does not completely decouple from the IR. As a consequence, naive scaling arguments break down, and the partition function develops "anomalous" logarithmic dependence on T for small T.

(a) Compute the partition function for the model (9) deformed by $\delta V(X,Y) = \epsilon X^2 Y^2$ analytically using Mathematica or some other symbolic software. This will give a horrible mess of hypergeometric functions. Expand it at small T and you should find something of the form

$$Z = Z_0 T^c \log\frac{\Lambda}{T} \tag{10}$$

up to corrections suppressed by positive powers of $\sqrt{T/\Lambda}$. Find the constants Z_0, c, Λ. The over all normalization Z_0 does not mean anything in classical statistical mechanics.

Mathematica will tell you that the integral

$$Z_V = \int_{-\infty}^{\infty} dX dY e^{-(aX^4 + bY^8 + \epsilon X^2 Y^2)/T}$$

is

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Clear[a]; a1 = 4 Integrate[e^(-(a X^4 + b Y^8 + ϵ X^2 Y^2)/T), {X, 0, ∞}, {Y, 0, ∞}, Assumptions → {T > 0, ϵ > 0, a > 0, b > 0}]
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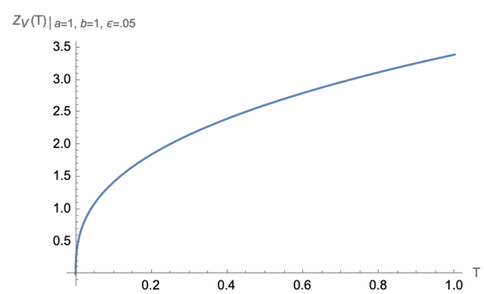
$$\frac{1}{48\, a^{7/4}\sqrt{b^3\, T}}\Bigg(192\, a^{3/2}\, b^{11/8}\, T^{7/8}\, \text{Gamma}\left[\frac{9}{8}\right] \text{Gamma}\left[\frac{5}{4}\right] \text{HypergeometricPFQ}\left[\left\{\frac{1}{8}, \frac{1}{8}, \frac{5}{8}\right\}, \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}, \frac{\epsilon^4}{64\, a^2\, b\, T}\right] - \epsilon\Bigg(6\, a\, b^{9/8}\, T^{5/8}\, \text{Gamma}\left[\frac{3}{8}\right] \text{Gamma}\left[\frac{3}{4}\right] \text{HypergeometricPFQ}\left[\left\{\frac{3}{8}, \frac{3}{8}, \frac{7}{8}\right\}, \left\{\frac{1}{2}, \frac{3}{4}, \frac{5}{4}\right\}, \frac{\epsilon^4}{64\, a^2\, b\, T}\right] + \epsilon\Bigg(-3\sqrt{a}\, b^{7/8}\, T^{3/8}\, \text{Gamma}\left[\frac{5}{8}\right] \text{Gamma}\left[\frac{5}{4}\right] \text{HypergeometricPFQ}\left[\left\{\frac{5}{8}, \frac{5}{8}, \frac{9}{8}\right\}, \left\{\frac{3}{4}, \frac{5}{4}, \frac{3}{2}\right\}, \frac{\epsilon^4}{64\, a^2\, b\, T}\right] + \left(b^5\, T\right)^{1/8} \epsilon\, \text{Gamma}\left[\frac{7}{8}\right] \text{Gamma}\left[\frac{7}{4}\right] \text{HypergeometricPFQ}\left[\left\{\frac{7}{8}, \frac{7}{8}, \frac{11}{8}\right\}, \left\{\frac{5}{4}, \frac{3}{2}, \frac{7}{4}\right\}, \frac{\epsilon^4}{64\, a^2\, b\, T}\right]\Bigg)\Bigg)\Bigg)$$

This function looks like:



The series expansion has a bit that goes like $\sqrt{T}\log T$ plus corrections of order $\sqrt{T}$, and a bit that goes like $Te^{\frac{\epsilon^4}{64a^2bT}}$. The latter is a very weird function. If it were $e^{-1/T}$ with a negative coefficient in the exponent, it would be easy to say that this is non-perturbatively small. With a positive but small coefficient (*i.e.* for small ϵ) it is essentially indistinguishable from T, as long as $T > 0$. Therefore it is subleading. If you plot each of these bits individually, you can see that the former is the part that matters.

(b) Using (10), compute the dimensionless quantities U/T and C. (Without the logarithmic dependence on T, these would be equal.) Check that in the strict limit $T \to 0$, you get the values for U/T and C that you would have guessed based on naive scaling arguments for $V \propto X^2Y^2$. Note that a logarithm varies more slowly than the $T^{1/2}$ corrections that we threw away.

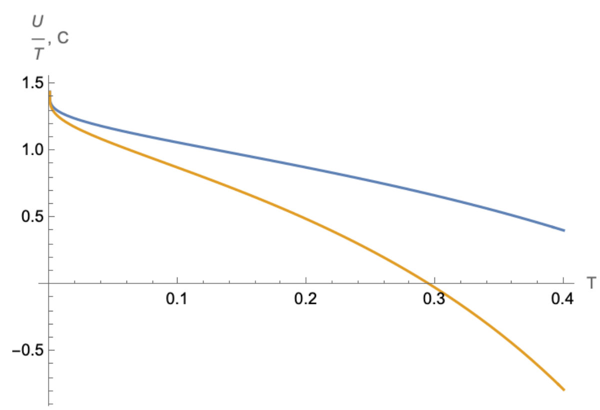
So $Z = Z_0 T^{1+\frac{1}{2}} \log T/\Lambda$ (don't forget the contribution from the two momentum integrals) and therefore

$$U/T = T\partial_T \log Z = \frac{3}{2} + \frac{1}{\log T/\Lambda} \tag{11}$$

while

$$C = \partial_T U = \frac{3}{2} + \frac{1}{\log T/\Lambda} - \frac{1}{\log^2 T/\Lambda}. \tag{12}$$

The naive answer is $Z \sim T^{1+1/2}$, using $Z_V \stackrel{?}{=} \int dXdYe^{-X^2Y^2\epsilon/T} = \sqrt{T/\epsilon}\int dxdye^{-x^2y^2}$ by scaling; this would work if the integral were actually well-defined without introducing some other scale. This gives $U/T = C = \frac{3}{2}$, and indeed both of the above functions do approach $\frac{3}{2}$ as $T \to 0$. The correct curves look like



(c) To what extent does the IR physics depend on the UV completion of the $V \propto X^2Y^2$ model? We could have started with $V = aX^8 + bY^8 + \epsilon X^2Y^2$ instead. This model would have different high-temperature physics. Redo part for this potential. You'll find an equally-horrendous, but different combination of hypergeometric functions. Which of the parameters Z_0, c, Λ are the same?

Only c is universal.

(d) The result of the previous part remains true for any other UV completion of the $V \propto X^2Y^2$ model, as long as $\delta V = \epsilon X^2Y^2$ remains a relevant deformation. In fact, we could equally well just take $V = \epsilon X^2Y^2$ and impose a hard cutoff on the X and Y integrals at some fixed values $|X| \leq X_0, |Y| \leq Y_0$

(this is like $V = X^n + Y^n$ with $n \to \infty$). Check that this again reduces to (10).

The answer is simpler:

$$Z_V^L \equiv \int_{-L}^{L} dX \int_{-L}^{L} dY e^{-\epsilon X^2 Y^2/T} = 4L^2 \texttt{HypergeometricPFQ}\left[\left\{\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{3}{2}, \frac{3}{2}\right\}, -\frac{L^4 \epsilon}{T}\right].$$

This has the simpler low-temperature expansion:

$$Z_V^L \sim -\sqrt{\frac{\pi T}{\epsilon}} \log \frac{T}{\epsilon L^4 \gamma} + \mathcal{O}(T^{3/2}) + e^{-L^4\epsilon/T}\mathcal{O}(T^2)$$

where γ is some irrelevant constant, and now the other term really is non-perturbatively small.

(e) In view of this apparent universality of (10) at low T, it is desirable to have a way of deriving it without having to take the detour involving the horrendous hypergeometric functions. Here is one way. We use the hard cutoff $|X| \leq L, |Y| \leq L$, so that the position-space factor is

$$Z_V(T, L) = \int_{-L}^{L} dX \int_{-L}^{L} dY e^{-X^2Y^2/T} \tag{13}$$

where we've set $\epsilon = 1$ by a choice of temperature units. A rescaling of the integration variables $(X, Y) \to (T^{1/4}X, T^{1/4}Y)$ shows that $Z_V(T, L) = \sqrt{T} F(T^{-1/4}L)$ for some function F of one variable. To find F, compute $L\partial_L Z_V$ directly from (13). By another suitable rescaling, show that $L\partial_L Z$ is finite and easily computable for $L^4/T \to \infty$. Infer from this the dependence on the cutoff L in the regime $T \ll L^4$ and thus the function F in this regime. This reproduces (10).

$$Z_V(T, L) = 4\int_0^L dX \int_0^L dY e^{-X^2Y^2/T} = \sqrt{T} F(T^{-1/4}L).$$

By the fundamental theorem of calculus,

$$L\partial_L Z_V = 4L \int_0^L dY e^{-L^2Y^2/T} \times 2$$

where the last factor of two comes from the place where the L derivative hits the upper limit of the Y integral. By scaling $y = L^2Y^2/T$ (so $dY = dy\sqrt{T}/L$) this is

$$L\partial_L Z_V = 8L\frac{\sqrt{T}}{L} \int_0^{L^2T^{-1/2}} dy e^{-y^2} = 8\sqrt{T}\left(\sqrt{\frac{\pi}{2}} + \mathcal{O}(e^{-L^4/T})\right).$$

Using $x\partial_x|_T = L\partial_L$, we have

$$x\partial x F(x) = T^{-1/2} L\partial_L Z_V = 4\sqrt{\pi} + \mathcal{O}(e^{-L^4/T}).$$

The solution of this ODE is $F(x) = c + 4\sqrt{\pi}\log x$, and therefore

$$Z_V(T, L) = \sqrt{\frac{T}{\epsilon}}(c + \sqrt{\pi}\log\frac{\epsilon L^4}{T}).$$

At the last step, I restored the ϵ by dimensional analysis. Since we don't care about the overall factor, we can get rid of the $\sqrt{\pi}$, and this is what we had above.

(f) We conclude that even when some kind of UV completion is required to give finite answers, the observable low-energy physics remains essentially independent of the UV completion. The infinite number of possible UV completions all flow in the IR to a partition function of the same form (10), with the details of the UV completion all lumped into a single scale parameter Λ. In fact, in the absence of other reference scales that can be used to fix a unit of temperature, the parameter Λ does not really label physically distinct models, since we can always choose units with $\Lambda = 1$. Equivalently, only dimensionless quantities (and relations between them) are physically meaningful. Examples of such dimensionless quantities are C and $u \equiv U/T$. Show that C and u obey a universal relation $C = f(u)$ with $f(u)$ independent of T and Λ, and thus independent of the UV completion of the X^2Y^2 model. In the same spirit, show that the function $g(u)$ in the flow equation $T\partial_T u = g(u)$ is independent of the UV completion.

A brute force way to do this is just to compute them both from $Z = Z_0 T\log T/\Lambda$ and find the answers in (11) and (12). Letting $L \equiv \frac{1}{\log T/\Lambda}$, we have

$$u = \frac{3}{2} + L, C = \frac{3}{2} + L - L^2$$

so $L = u - \frac{3}{2}$ and

$$C = -u^2 + 3u - \frac{3}{2} \equiv f(u).$$

Similarly,

$$T\partial_T u = -\frac{1}{\log T/\Lambda} = -L^2 = -\left(u - \frac{3}{2}\right)^2 \equiv g(u).$$

(g) Show that on the other hand $f(u)$ and $g(u)$ *do* depend on the IR part of the potential, for example by comparing the IR potential $V = X^2Y^2$ considered above to another IR potential such as $V = X^6Y^6$.

If instead we used $\delta V = \epsilon X^6 Y^6$, we would find in part 5e instead

$$Z_V(T, L) = T^{1/6} F(T^{-1/12} L)$$

and

$$x\partial_x F(x) = T^{-\frac{1}{6}} L \partial_L Z_V = 8 \int_0^{L^2 T^{-\frac{1}{6}}} dy e^{-y^6} = 8\Gamma(7/6) + \mathcal{O}(e^{-L^6/\sqrt{T}}).$$

Therefore, in the limit $T \ll L^{12}$, the solution is

$$Z_V = T^{1/6}(c + 8\Gamma(7/6)\log(T^{\frac{-1}{12}} L))$$

and therefore

$$Z = Z_0 T^{1+\frac{1}{6}} \log T/\Lambda$$

and

$$u = U/T = T\partial_T \log Z = \frac{7}{6} + \frac{1}{\log T/\Lambda} = \frac{2}{3} + L \tag{14}$$

while

$$C = \partial_T U = \frac{7}{6} + \frac{1}{\log T/\Lambda} - \frac{1}{\log^2 T/\Lambda} = \frac{7}{6} + L - L^2. \tag{15}$$

These satisfy $L = u - \frac{7}{6}$, so

$$C = u - \left(u - \frac{7}{6}\right)^2 = f(u)$$

and $T\partial_T u = -L^2 = -\left(u - \frac{7}{6}\right)^2 = g(u)$ are indeed different.