University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2022
Assignment 7 – Solutions

Due 11:59pm Monday, February 21, 2022

Thanks in advance for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Yukawa couplings in QED.** [optional] Consider adding to QED an additional scalar field of (physical) mass m, coupled to the electron by

$$L_Y = \lambda\phi\bar{\psi}\psi.$$

Verify that the divergent contribution to the electron wavefunction renormalization factor Z_2 from a virtual ϕ equals the divergent contribution to the QED vertex Z_1 from the one loop correction to the vertex with a virtual ϕ. For an added challenge, verify that the finite parts agree as well.

Since we are only worried about the UV divergences here, in the vertex correction, one only need attend to the ℓ^2 term in the numerator of the integrand. In dim reg, the divergent parts are

$$\delta_1^{div} = 2\lambda^2 \int_0^1 dx \int_0^{1-x} dy \frac{\epsilon - 2}{D} \frac{D}{2} \frac{\Gamma(2 - D/2)}{(4\pi)^{D/2}\Gamma(3)} \bar{\mu}^\epsilon \Delta^{D/2-2}.$$

and

$$\delta_2^{div} = \left(\partial_{\not p}\delta\Sigma|_{\not p = m}\right)^{div} = -\lambda^2 \int_0^1 dx (1-x) \frac{\Gamma(2 - D/2)}{(4\pi)^{D/2}\Gamma(2)} \bar{\mu}^\epsilon \Delta^{D/2-2}$$

Using the identity $\Gamma(1+x) = x\Gamma(x)$, we have $\frac{D}{2\Gamma(3)} = \frac{1}{\Gamma(2)}$ and $\delta_1^{div} = \delta_2^{div}$ as $D \to 4$.

For purposes of matching the finite parts, some advice: we can put the electron lines on shell and sandwich between spinors satisfying the equations of motion (as we did for the QED vertex correction), and also set the incoming photon momentum $q = p' - p = 0$.

When using dimensional regularization, to get the finite parts to agree it is necessary to continue all appearances of $D = 4$ to D dimensions. In particular, the number of gamma matrices should be D, and in particular one must use the identity:

$$\not\ell \gamma^\mu \not\ell = \frac{\ell^2}{D}\gamma_\nu\gamma^\mu\gamma^\nu = -\frac{D-2}{D}\ell^2\gamma^\mu.$$

2. **Another consequence of unitarity of the S matrix.**

(a) Show that unitarity of S, $S^\dagger S = \mathbb{1} = SS^\dagger$, implies that the transition matrix is *normal*:

$$\mathcal{T}\mathcal{T}^\dagger = \mathcal{T}^\dagger\mathcal{T} \ . \tag{1}$$

(b) What does this mean for the amplitudes $\mathcal{M}_{\alpha\beta}$ (defined as usual by $\mathcal{T}_{\alpha\beta} = \not{\delta}(p_\alpha - p_\beta)\mathcal{M}_{\alpha\beta}$)?

Taking matrix elements, and inserting a resolution of the identity, this says

$$\sum_\beta \mathcal{T}_{\alpha\beta}\mathcal{T}^\star_{\gamma\beta} = \sum_\beta \mathcal{T}^\star_{\beta\alpha}\mathcal{T}_{\gamma\alpha}.$$

In terms of $\mathcal{M}$, it says

$$\sum_\beta \mathcal{M}_{\alpha\beta}\mathcal{M}^\star_{\gamma\beta}\not{\delta}_{\alpha\beta} = \sum_\beta \mathcal{M}^\star_{\beta\alpha}\mathcal{M}_{\beta\gamma}\not{\delta}_{\alpha\beta}$$

as long as $\alpha = \gamma$. The diagonal entries will be useful below:

$$\sum_\beta \not{\delta}_{\alpha\beta}|\mathcal{M}_{\alpha\beta}|^2 = \sum_\beta \not{\delta}_{\alpha\beta}|\mathcal{M}_{\beta\alpha}|^2. \tag{2}$$

(c) The probability of a transition from α to β is

$$P_{\alpha\to\beta} = |S_{\beta\alpha}|^2 = VT\not{\delta}(p_\alpha - p_\beta)|\mathcal{M}_{\alpha\beta}|^2$$

which is IR divergent. More useful is the transition rate per unit time per unit volume:

$$\Gamma_{\alpha\to\beta} \equiv \frac{P_{\alpha\to\beta}}{VT}.$$

Show that the the total decay rate of the state α is

$$\Gamma_\alpha \equiv \int d\beta\Gamma_{\alpha\to\beta} = 2\mathrm{Im}\,\mathcal{M}_{\alpha\alpha}.$$

(d) Consider an ensemble of states p_α evolving according to the evolution rule

$$\partial_t p_\alpha = -p_\alpha\Gamma_\alpha + \int d\beta p_\beta \Gamma_{\beta\to\alpha}. \tag{3}$$

$S[p] \equiv -\int d\alpha p_\alpha \ln p_\alpha$ is the Shannon entropy of the distribution. Show that

$$\frac{dS}{dt} \geq 0$$

as a consequence of (1). This is a version of the Boltzmann H-theorem.

$$\frac{dS}{dt} = -\sum_\alpha \dot{p}_\alpha (\ln p_\alpha + 1) = -\sum_\alpha \dot{p}_\alpha \ln p_\alpha \tag{4}$$

$$= -\sum_\alpha \left(-p_\alpha \sum_\beta \Gamma_{\alpha\to\beta} + \sum_\beta p_\beta \Gamma_{\beta\to\alpha} \right) \ln p_\alpha \tag{5}$$

$$\overset{\text{part b}}{=} -\sum_{\alpha\beta} \delta(k_\beta - k_\alpha) |\mathcal{M}_{\beta\alpha}|^2 (p_\beta - p_\alpha) \ln p_\alpha \tag{6}$$

$$\overset{\text{relabel in second term}}{=} -\sum_{\alpha\beta} \delta(k_\beta - k_\alpha) |\mathcal{M}_{\beta\alpha}|^2 \left(p_\beta \log\left(p_\alpha / p_\beta \right) \right). \tag{7}$$

In the first line, the second term drops out because $0 = \partial_t(1) = \partial_t(\sum_\alpha p_\alpha)$. In the step labelled 'part b', we used (2) to eliminate

$$\sum_\beta \Gamma_{\alpha\to\beta} = \sum_\beta \Gamma_{\beta\to\alpha} = \sum_\beta \not{\delta}_{\alpha\beta} |\mathcal{M}_{\beta\alpha}|^2.$$

Actually, we could have done this manipulation in the expression for $\dot{p}_\alpha$, before saying anything about $\dot{S}$, and it shows that :

$$\dot{p}_\alpha = \sum_\beta (p_\alpha - p_\beta) \Gamma_{\beta\to\alpha}.$$

This fact that the same rate governs the incoming and outgoing probabilities is usually called *reversibility* of the dynamics.

The right hand side of (7) is a *relative entropy*, which is positive (for more on this, see this physics 239 class). To see this explicitly: for $x \in [0, 1]$, $\log x \geq x - 1$ (this is Jensen's inequality). Applying this to $x = p_\alpha / p_\beta$ gives $p_\alpha \log p_\alpha / p_\beta \geq p_\alpha - p_\beta$ and therefore

$$\frac{dS}{dt} = -\sum_{\alpha\beta} \delta(k_\beta - k_\alpha) |\mathcal{M}_{\beta\alpha}|^2 \left(p_\beta \log\left(p_\alpha / p_\beta \right) \right) \tag{8}$$

$$\geq -\sum_{\alpha\beta} \delta(k_\beta - k_\alpha) |\mathcal{M}_{\beta\alpha}|^2 \left(p_\alpha - p_\beta \right) = 0 \tag{9}$$

where at the last step the integrand is odd under $\alpha \leftrightarrow \beta$.

(e) [Bonus] Notice that we are doing something weird in the previous part by using classical probabilities. This is a special case; more generally, we should describe such an ensemble by a density matrix $\rho_{\alpha\beta}$. Generalize the result of the previous part appropriately.

The generalization of the Shannon entropy $S(p)$ is the von Neumann entropy $S[\rho] = -\text{tr}\rho \log \rho$. The tricky thing here is figuring out how ρ should evolve in time. If we let it evolve unitarily, $\mathbf{i}\dot{\rho} = [H, \rho]$ for some hermitian H, then $\dot{S} = 0$.

But actually, we need not make too much assumption about $\dot{\rho}$ to answer this question: Any density matrix $\rho = \rho^\dagger$ has a spectral representation, $\rho = \sum_\alpha p_\alpha |\alpha\rangle\langle\alpha|$, and the eigenvalues are probabilities. Whatever the evolution of ρ, if $\dot{\rho}$ depends only on ρ at the current time, then these eigenvalues must evolve according to the master equation (3). And then the calculation of the previous part follows.

The reason this '$\dot{\rho}(t)$ depends only on $\rho(t)$' is an assumption is that in general it can depend on the whole past history of the state – the environment can have a memory. The name for the equation for $\dot{\rho}$ in the case where the environment is forgetful is the Lindblad equation.

3. **An application of effective field theory in quantum mechanics.**

[I learned this example from Z. Komargodski.]

Consider a model of two canonical quantum variables ($[\mathbf{x}, \mathbf{p}_x] = \mathbf{i} = [\mathbf{y}, \mathbf{p}_y], 0 = [\mathbf{x}, \mathbf{p}_y] = [\mathbf{x}, \mathbf{y}]$, etc) with Hamiltonian

$$\mathbf{H} = \mathbf{p}_x^2 + \mathbf{p}_y^2 + \lambda \mathbf{x}^2 \mathbf{y}^2.$$

(This is similar to the degenerate limit of the model studied in lecture with two QM variables where both natural frequencies are taken to zero.)

(a) Based on a semiclassical analysis, would you think that the spectrum is discrete or continuous?

The potential has flat directions along the coordinate axes, $\{x = 0\} \cup \{y = 0\}$. This means there are unbounded classical orbits, which suggests that the spectrum should be continuous. This conclusion is in fact wrong. (An excuse for discounting it is that the set of initial conditions which follow unbounded orbits have measure zero.)

(b) Study large, fixed x near $y = 0$. We will treat x as the slow (= low-energy) variable, while y gets a large restoring force from the background x value. Solve the y dynamics, and find the groundstate energy as a function of x:

$$V_{\text{eff}}(x) = E_{\text{g.s. of y}}(x).$$

If we treat x as a constant, the hamiltonian for y is a harmonic oscillator problem. The groundstate energy is

$$V_{\text{eff}}(x) = E_{\text{g.s. of y}}(x) = \sqrt{\lambda}|x|$$

(c) [Bonus] Presumably you did the previous part using your knowledge of the spectrum of the harmonic oscillator. Redo the previous part using path integral methods.

We can also do it using path integrals. Let's do it in euclidean time. The Lagrangian which gives $H = p^2 + \omega^2 x^2$ is $L = \frac{1}{4}\dot{x}^2 - \frac{\omega^2}{4}x^2$, with $\omega^2 = 4\lambda x^2$. The integral we need is

$$\int [Dy] \exp(-\int dt y M y) = \det M^{-1/2} = e^{-\frac{1}{2}\mathrm{tr}\log M}$$

with $M \equiv (-\frac{\partial^2}{4} + \lambda x^2)$. This gives a correction to the effective action for x

$$e^{-\delta S_{\text{eff}}[x]} = e^{-\frac{1}{2}\mathrm{tr}\log M}.$$

If we treat x as constant, and consider a time interval T, this corrects the effective potential by

$$V_{\text{eff}}(x) = +\frac{1}{2}\mathrm{tr}\log M/T = +\frac{1}{2}\int_{-\Lambda}^{\Lambda} đ\omega \log\left(\frac{\omega^2}{4} + \lambda x^2\right) \tag{10}$$

$$= \frac{1}{2}\left(-2\Lambda(2 + \log 4 - 2\log\Lambda) + 2\sqrt{\lambda}|x| + \mathcal{O}(1/\Lambda)\right). \tag{11}$$

We need to regulate the frequency integral and ignore the meaningless additive constant, but we get the same answer as with the canonical method. Note that doing this calculation with non-constant x, we can do a derivative expansion and some additional terms involving derivatives of x will also be produced. These don't change the conclusion below about the spectrum.

(d) The result for $V_{\text{eff}}(x)$ is not analytic in x at $x = 0$. Why?

At $x = 0$, y becomes massless (i.e. it is a spring whose natural frequency goes to zero there). Integrating out massless degrees of freedom produces singularities in the effective action.

(e) Is the spectrum of the resulting 1d model with

$$\mathbf{H}_{\text{eff}} = \mathbf{p}_x^2 + V_{\text{eff}}(\mathbf{x})$$

discrete? Is this description valid in the regime that matters for the semi-classical analysis?

[Bonus: determine the spectrum of $\mathbf{H}_{\text{eff}}$.]

The potential $V \sim |x|$ bounds the trajectories and has a discrete spectrum. Integrating out y is a better approximation at larger $|x|$, which is where the dangerous flat directions occur. That is: this approximation is valid outside of a compact region of field space near $x = y = 0$ in which the potential is bounded below. Such a region cannot produce a continuum in the spectrum.

The actual spectrum of the absolute value potential is fun. The solutions of the Schrödinger problem (we can rescale x to get rid of the constant prefactor in the potential) $\psi(x) = \psi_>(x)\theta(x) + \psi_<(x)\theta(-x)$ satisfy

$$\begin{cases} \left(-\partial_x^2 + (x - E)\right)\psi_> = 0, & x > 0 \\ \left(-\partial_x^2 + (-x - E)\right)\psi_< = 0, & x > 0 \end{cases}.$$

The solutions for $x > 0$ are the two Airy functions

$$\psi_>(x) = a_> \mathrm{Ai}(x - E) + b_> \mathrm{Bi}(x - E)$$

of which the second blows up at large argument and hence cannot be normalized so we must set $b_> = 0$. Similarly, for $x < 0$, we have

$$\psi_<(x) = a_< \mathrm{Ai}(-x - E) + b_< \mathrm{Bi}(-x - E)$$

and again we must set $b_< = 0$. Since the potential has finite measure near $x = 0$ (i.e. no delta function) the wavefunction and its first derivative must be continuous at $x = 0$ and we have

$$\begin{array}{lcl} \psi_>(0) = \psi_<(0) & \implies & a_> \mathrm{Ai}(-E) = a_< \mathrm{Ai}(-E) \\ \psi'_>(0) = \psi'_<(0) & \implies & a_> \mathrm{Ai}'(-E) = -a_< \mathrm{Ai}'(-E) \end{array} \tag{12}$$

which means either $a_> = a_< = 0$ OR $\mathrm{Ai}(-E) = 0$ OR $\mathrm{Ai}'(-E) = 0$. This means that the boundstates occur at zeros of the airy function or its derivative:

$$\{\text{boundstate energies }\} \propto \{E|\mathrm{Ai}(-E) = 0 \text{ or } \mathrm{Ai}'(-E) = 0\}.$$