University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2022
Assignment 8 – Solutions

Due 11:59pm Monday, February 28, 2022

Thanks in advance for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Gauge theory brain-warmers.**

   (a) Show that the *adjoint* representation matrices[1]

$$\left(T^{A}\right)_{BC} \equiv -\mathbf{i} f_{ABC}$$

furnish a $\dim \mathsf{G}$-dimensional representation of the Lie algebra

$$[T^A, T^B] = \mathbf{i} f_{ABC} T^C \quad .$$

Hint: commutators satisfy the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

The structure constants f_C^{AB} are part of the definition of the Lie algebra – in *any* representation, the generators satisfy $[T^A, T^B] = \mathbf{i} f_C^{AB} T^C$. This is a property of the algebra, not of any particular representation. The Jacobi identity follows from this fact, by taking the commutator of the BHS with T^D. Reshuffling this identity gives the desired equation (up to a sign which may be flipped by redefining $T \to -T$).

More abstractly, the operation $B \to \mathrm{ad}_A(B) \equiv [A, B]$ is called the *adjoint* action of A on B. The Jacobi identity is then the statement that $\mathrm{ad}_A \mathrm{ad}_B(C) - \mathrm{ad}_B \mathrm{ad}_A(C) = \mathrm{ad}_{[A,B]}(C)$, *i.e.*$[\mathrm{ad}_A, \mathrm{ad}_B] = \mathrm{ad}_{[A,B]}$. This is the statement that the map $A \to \mathrm{ad}_A$ preserves the Lie algebra, and hence gives a representation, which is inevitably called the adjoint representation. In terms of the generators of an arbitrary representation, $\mathrm{ad}_{T^A} T^B = [T^A, T^B] = \mathbf{i} f_{ABC} T^C$, we find an expression for the adjoint generators, which is indeed $\left(T^A_{\text{adj}}\right)_{BC} = \mathbf{i} f_{ABC}$ with the opposite sign from what I said.

[1] Thanks to Simon Martin for help with the signs.

(b) [optional, added on Wednesday 2022-02-23] Show that if $(T_A)_{ij}$ are generators of a Lie algebra in some unitary representation R, then so are $-(T_A)^\star_{ij}$. Convince yourselves that these are the generators of the complex conjugate representation $\bar{R}$.

We have $[T_A, T_B] = \mathbf{i} f_{ABC} T_C$, so $([T_A, T_B])^\star = -\mathbf{i} f_{ABC} T^\star_C$ (the structure constants are real for a unitary rep) so $[T^\star_A, T^\star_B] = -\mathbf{i} f_{ABC} T^\star_C$, so $[-T^\star_A, -T^\star_B] = \mathbf{i} f_{ABC}(-T^\star_C)$.

The representation operators in the rep R are $e^{\mathbf{i}\alpha^A T^A}$, with α^A real and T^A hermitian (if R is a unitary representation). In the rep $\bar{R}$, they are $e^{-\mathbf{i}\alpha^A (T^A)^\star}$, so the generators in this rep are indeed $-(T^A)^\star_{ij}$.

(c) [optional, added on Wednesday 2022-02-23] Show that in a basis of Lie algebra generators where $\mathrm{tr} T^A T^B = \lambda \delta^{AB}$, the structure constants f_{ABC} are completely antisymmetric.

Start from the Lie algebra $[T^A, T^B] = \mathbf{i} f^{ABC} T^C$, multiply the BHS by T^D on the right and take the trace:

$$\lambda \mathbf{i} f^{ABD} = \mathrm{tr}[T^A, T^B] T^D = \mathrm{tr} T^A T^B T^D - \mathrm{tr} T^B T^A T^D$$

and now use cyclicity of the trace, to show that this is the same as, *e.g.* $\mathrm{tr}[T^D, T^A] T^B$.

(d) From the transformation law for A, show that the non-abelian field strength transforms in the adjoint representation of the gauge group.

Mindlessly plugging in, we have

$$\begin{aligned}
F^A_{\mu\nu} \mapsto\ & \partial_\mu \left(A^A_\nu + \partial_\nu \lambda^A - f_{ABC} \lambda^B A^C_\nu \right) - (\mu \leftrightarrow \nu) \\
& + f_{ABC} \left(A^B_\mu + \partial_\mu \lambda^B - f_{BDE} \lambda^D A^E_\mu \right) \left(A^C_\nu + \partial_\nu \lambda^C - f_{CFG} \lambda^F A^G_\nu \right) \\
=\ & F^A_{\mu\nu} - f_{ABC} \lambda^B \partial_\mu A^C_\nu + f_{ABC} \lambda^B \partial_\nu A^C_\mu - f_{ABC} f_{CFG} \lambda^F A^B_\mu A^G_\nu - f_{ABC} f_{BDE} \lambda^D A^E_\mu A^C_\nu \quad (1) \\
=\ & F^A_{\mu\nu} - f_{ABC} \lambda^B \partial_\mu A^C_\nu + f_{ABC} \lambda^B \partial_\nu A^C_\mu - \lambda^D A^E_\mu A^C_\nu \left(f_{ABC} f_{BDE} + f_{AEB} f_{BDC} \right) \quad (2) \\
=\ & F^A_{\mu\nu} - f_{ABC} \lambda^B \partial_\mu A^C_\nu + f_{ABC} \lambda^B \partial_\nu A^C_\mu - \lambda^D A^E_\mu A^C_\nu f_{ADB} f_{BEC} \quad (3) \\
=\ & F^A_{\mu\nu} - f_{ABC} \lambda^B \partial_\mu A^C_\nu + f_{ABC} \lambda^B \partial_\nu A^C_\mu - \lambda^B A^D_\mu A^E_\nu f_{ABC} f_{CDE} \quad (4) \\
=\ & F^A_{\mu\nu} - \lambda^B f_{ABC} F^C_{\mu\nu}. \quad (5)
\end{aligned}$$

Everywhere we ignored $\mathcal{O}(\lambda^2)$ terms. At step (2) we used the Jacobi identity. At steps (1) and (3) we relabelled dummy indices.

(e) Show that

$$\mathrm{tr} F \wedge F = d\mathrm{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$

Write out all the indices I've suppressed.

On the LHS, we find $\mathrm{tr} F \wedge F = \mathrm{tr} dA \wedge dA + 2\mathrm{tr} dA \wedge A \wedge A + \mathrm{tr} A \wedge A \wedge A \wedge A$ (using the fact that two-forms are commutative). The last term vanishes by cyclicity of the trace:

$$\mathrm{tr} A^4 \equiv \mathrm{tr} A \wedge A \wedge A \wedge A \equiv A^a \wedge A^b \wedge A^c \wedge A^d \mathrm{tr} T^a T^b T^c T^d \tag{6}$$

$$\overset{\text{cyclicity}}{=} A^a \wedge A^b \wedge A^c \wedge A^d \mathrm{tr} T^d T^a T^b T^c \tag{7}$$

$$\overset{\{A^3, A\}=0}{=} -A^d \wedge A^a \wedge A^b \wedge A^c \mathrm{tr} T^d T^a T^b T^c \tag{8}$$

$$\overset{\text{relabel}}{=} -A^a \wedge A^b \wedge A^c \wedge A^d \mathrm{tr} T^a T^b T^c T^d = -\mathrm{tr} A^4. \tag{9}$$

On the RHS we get

$$d\mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) = \mathrm{tr} dA \wedge dA + A \wedge d^2 A \tag{10}$$

$$+ 2/3\left(dA \wedge A \wedge A + A \wedge dA \wedge A + A \wedge A \wedge dA\right) \tag{11}$$

$$= \mathrm{tr} dA \wedge dA + 2 dA \wedge A \wedge A \tag{12}$$

using $d^2 = 0$ and again the fact that a 2-form (such as dA) is commutative.

(f) [Bonus] If you are feeling under-employed, find ω_{2n-1} such that $\mathrm{tr} F^n = d\omega_{2n-1}$.

2. **The field of a magnetic monopole.**

We saw that $F = dA$ implies (when A is smooth) that $dF = 0$, which means no magnetic charge. If A is singular, dF can be nonzero. Moreover, by a gauge transformation we can move the singularity around and hide it.

A magnetic monopole of magnetic charge g is defined by the condition that $\int_{S^2} F = g$, where S^2 any sphere surrounding the monopole. If the system is spherically symmetric, we can write

$$F = \frac{g}{4\pi} d\cos\theta d\varphi.$$

(In this problem, we'll work on a sphere at fixed distance from the monopole.)

(a) Show that the vector potential

$$A_N = \frac{g}{4\pi}\left(\cos\theta - 1\right) d\varphi$$

gives the correct $F = dA$. Show that it is a well-defined one-form on the sphere except at the south pole $\theta = \pi$.

(b) Show that the one-form

$$A_S = \frac{g}{4\pi}(\cos\theta + 1)\,d\varphi$$

also gives the correct $F = dA$. Show that it is well-defined except at the north pole $\theta = 0$.

(c) Near the equator both $A_{N,S}$ are well-defined. Show that *as long as* $eg \in 2\pi\mathbb{Z}$, these two one-forms differ by a gauge transformation

$$A_S - A_N = \frac{1}{\mathbf{i}e} g^{-1}(\theta,\varphi) dg(\theta,\varphi)$$

for $g(\theta,\varphi)$ a $\mathsf{U}(1)$-valued function on the sphere, well-defined away from the poles.

Zee page 249. The required $g(\varphi) = e^{\mathbf{i}2\frac{eg}{4\pi}\varphi}$, which is single-valued $g(0) = g(2\pi)$ only under the stated condition (which is Dirac quantization of magnetic charge).