University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2022
Assignment 9 – Solutions

Due 11:59pm Monday, March 7, 2022

Thanks in advance for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

---

1. **Abrikosov-Nielsen-Oleson vortex string.**

Consider the Abelian Higgs model in $D = 3 + 1$:

$$\mathcal{L}_h \equiv -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}|D_\mu\phi|^2 - V(|\phi|)$$

where ϕ is a scalar field of charge q whose covariant derivative is $D_\mu\phi = (\partial_\mu - \mathbf{i}qeA_\mu)\,\phi$, and let's take

$$V(|\phi|) = \frac{\kappa}{2}(|\phi|^2 - v^2)^2$$

for some couplings κ, v. Here we are going to do some interesting classical field theory. Set $q = 1$ for a bit.

(a) Consider a configuration that is independent of x^3, one of the spatial coordinates, and static (independent of time). Show that its energy density (energy per unit length in x^3) is

$$U = \int d^2x\ \left(\frac{1}{2}F_{12}^2 + \frac{1}{2}|D_i\phi|^2 + V(|\phi|)\right).$$

Note that if the x^3 direction is non-compact (as opposed to a circle), then we can set $A_3 = 0$ by a gauge transformation depending on x^3 without changing anything.

The easiest way to do this is to do the Legendre transform of the action.

(b) [optional, but used crucially below] Consider the special case where $\kappa = \kappa_0 = \left(\frac{eq}{2}\right)^2$. Assuming that the integrand falls off sufficiently quickly at large $x^{1,2}$, show that

$$U_{\kappa=1} = \int d^2x\ \left(\frac{1}{2}\left(F_{12} + \sqrt{\kappa}\left(|\phi|^2 - v^2\right)\right)^2 + \frac{1}{4}|D_i\phi + \mathbf{i}\epsilon_{ij}D_j\phi|^2 + \sqrt{\kappa}v^2F_{12} - \frac{1}{2}\mathbf{i}\epsilon_{k\ell}\partial_k\left(\phi^\star D_\ell\phi\right)\right)$$

The key identity we need to show is:

$$\frac{1}{4}|D_i\phi + \mathbf{i}\epsilon_{ij}D_j\phi|^2 - \frac{1}{2}\mathbf{i}\epsilon_{k\ell}\partial_k\left(\phi^\star D_\ell\phi\right) = \frac{1}{2}|D_i\phi|^2 - \frac{eq}{2}F_{12}|\phi|^2. \qquad (1)$$

The LHS is

$$\frac{1}{4}|D_i\phi + \mathbf{i}\epsilon_{ij}D_j\phi|^2 - \frac{1}{2}\mathbf{i}\epsilon_{k\ell}\partial_k\left(\phi^\star D_\ell\phi\right) \tag{2}$$

$$= \frac{1}{4}|D_1\phi + \mathbf{i}D_2\phi|^2 + \frac{1}{4}|D_2\phi - \mathbf{i}D_1\phi|^2 - \frac{\mathbf{i}}{2}\left(\partial_1(\phi^\star D_2\phi) - \partial_2(\phi^\star D_1\phi)\right). \tag{3}$$

Note that since $\phi^\star D_i\phi$ is neutral, the ordinary derivatives are the same as covariant derivatives: $\partial_i\left(\phi^\star D_j\phi\right) = D_i(\phi^\star D_j\phi)$, and then we can use the fact that the covariant derivative satisfies the product rule to write the LHS as:

$$\frac{1}{2}|D_i\phi|^2 + \frac{\mathbf{i}}{2}\left((D_1\phi)^\star D_2\phi - (D_2\phi)^\star D_1\phi\right)(1-1) + \frac{\mathbf{i}}{2}\phi^\star[D_1, D_2]\phi \tag{4}$$

$$= \frac{1}{2}|D_i\phi|^2 - \frac{eq}{2}F_{12}|\phi|^2. \tag{5}$$

(c) The first two terms in the energy density of the previous part are squares and hence manifestly positive, so setting to zero the things being squared will minimize the energy density. Show that the resulting first-order equations (they are called BPS equations after people with those initials, Bogolmonyi, Prasad, Sommerfeld)[1]

$$0 = \left(D_i + \mathbf{i}\epsilon_{ij}D_j\right)\phi, \quad F_{12} = -|\phi|^2 + v^2$$

are solved by $(x^1 + \mathbf{i}x^2 \equiv re^{\mathbf{i}\varphi})$

$$\phi = e^{\mathbf{i}n\varphi}f(r), \quad A_1 + \mathbf{i}A_2 = -\mathbf{i}e^{\mathbf{i}\varphi}\frac{a(r)-n}{r}$$

if

$$f' = \frac{a}{r}f, \; a' = r(f^2 - v^2)$$

with boundary conditions

$$a \to 0, f \to v + \mathcal{O}\left(e^{-mr}\right), \quad \text{at } r \to \infty \tag{6}$$

$$a \to n + \mathcal{O}(r^2), f \to r^n(1 + \mathcal{O}(r^2)), \quad \text{at } r \to 0.$$

(For other values of κ, the story is not as simple, but there is a solution with the same qualitative properties. See for example §3.3 of E. Weinberg, *Classical solutions in Quantum Field Theory.*)

[1]Let's set $\kappa = 1$ for this discussion; it does not affect the qualitative conclusions.

(d) The second BPS equation and (6) imply that all the action (in particular the support of F_{12}) is localized near $r = 0$. Evaluate the magnetic flux through the $x^1 - x^2$ plane, $\Phi \equiv \int B \cdot da$ in the vortex configuration labelled by n. Show that the energy density is $U = \frac{v^2}{2} \cdot \Phi$.

To evaluate the flux, one method is to write

$$A_x dx + A_y dy = A_z dz + A_{\bar{z}} d\bar{z}$$

with $z = re^{\mathbf{i}\varphi}, \bar{z} = re^{-\mathbf{i}\varphi}$, so that

$$A_z = \frac{1}{2}(A_x - \mathbf{i}A_y) = \frac{\mathbf{i}}{2}e^{-\mathbf{i}\varphi}\frac{a(r) - n}{r}, A_{\bar{z}} = \frac{1}{2}(A_x + \mathbf{i}A_y) = -\frac{\mathbf{i}}{2}e^{\mathbf{i}\varphi}\frac{a(r) - n}{r}.$$

By Stokes theorem, $\int_X B = \int_X F = \int_X dA = \oint_{\partial X} A$ where X is a big disk about the origin and so ∂X is a big circle far enough from the origin that we can ignore the $a(r)$ in A. Along this contour, $dz = \mathbf{i}e^{\mathbf{i}\varphi} r d\varphi, d\bar{z} = -\mathbf{i}e^{-\mathbf{i}\varphi} r d\varphi$, *i.e.* we can ignore the dr part. Then

$$\oint_X A = \oint (dz A_z + d\bar{z} A_{\bar{z}}) = \int_0^{2\pi} r d\varphi \left(\mathbf{i}e^{\mathbf{i}\varphi} A_z - \mathbf{i}e^{-\mathbf{i}\varphi} A_{\bar{z}} \right) \tag{7}$$

$$\overset{r \gg m_v}{=} \int_0^{2\pi} d\varphi \left(\frac{1}{2}\mathbf{i}^2 e^{\mathbf{i}\varphi} e^{-\mathbf{i}\varphi} (-n) + \frac{1}{2}(-\mathbf{i})^2 e^{-\mathbf{i}\varphi} e^{\mathbf{i}\varphi} (-n) \right) = 2\pi n \ . \tag{8}$$

(e) Show that the previous result for the flux follows from demanding that the two terms in $D_i\phi$ cancel at large r so that

$$D_i \phi \overset{r \to \infty}{\to} 0 \tag{9}$$

faster than $1/r$. Solve (11) for A_i in terms of ϕ and integrate $\int d^2 x F_{12}$.

Zee page 307 (with charge q):

$$A_i \overset{r \to \infty}{\to} -\frac{\mathbf{i}}{qe}\frac{1}{\rho^2}\phi^\star \partial_i \phi = \frac{1}{qe}\partial_i \varphi.$$

Therefore

$$\Phi = \int d^2 x F_{12} = \oint dx^i A_i = \frac{2\pi}{qe}.$$

(f) Argue that a single vortex (string) in the ungauged theory (with global $\mathsf{U}(1)$ symmetry)

$$\mathcal{L} = |\partial \phi|^2 + V(|\phi|)$$

does not have finite energy per unit length. By a vortex, I mean a configuration where $\phi \overset{r \to \infty}{\to} ve^{\mathbf{i}\varphi}$, where $x^1 + \mathbf{i}x^2 = re^{\mathbf{i}\varphi}$.

The kinetic energy density is

$$\partial_\mu \phi \partial^\mu \phi = |\partial_r \phi|^2 + r^{-2} |\partial_\varphi \phi|^2 = ... + r^{-2} v^2 n^2$$

so the energy per unit length is

$$U \geq \int_a^L dr r \frac{1}{r^2} v^2 n^2 = v^2 n^2 \ln \frac{L}{a}$$

where L is the size of the box and a is the short distance cutoff.

(g) Consider now the case where the scalar field has charge q. (Recall that in a superconductor made by BCS pairing of electrons, the charged field which condenses has electric charge two.) Show that the magnetic flux in the core of the minimal ($n = 1$) vortex is now (restoring units) $\frac{hc}{qe}$.

2. **BPS conditions from supersymmetry.** [bonus!] What's special about $\kappa = \kappa_0$? For one thing, it is the boundary between type I and type II superconductors (which are distinguished by the size of the vortex core). More sharply, it means the mass of the scalar equals the mass of the vector, at least classically. Moreover, in the presence of some extra fermionic fields, the model with this coupling has an additional symmetry mixing bosons and fermions, namely supersymmetry. This symmetry underlies the special features we found above. Here is an outline (you can do some parts without doing others) of how this works. The logic in part (c) underlies a lot of the progress in string theory since the mid-1990s. Please do not trust my numerical factors.

(a) Add to $\mathcal{L}_h$ a charged fermion Ψ (partner of ϕ) and a neutral Majorana fermion λ (partner of A_μ):

$$\mathcal{L}_f = \frac{1}{2} \mathbf{i} \bar{\Psi} \not{D} \Psi + \mathbf{i} \bar{\lambda} \not{D} \lambda + \bar{\lambda} \Psi \phi + h.c..$$

Consider the transformation rules

$$\delta_\epsilon A_\mu = \mathbf{i} \bar{\epsilon} \gamma_\mu \lambda, \delta_\epsilon \Psi = D_\mu \phi \gamma^\mu \epsilon, \quad \delta_\epsilon \phi = -\mathbf{i} \bar{\epsilon} \Psi, \delta_\epsilon \lambda = -\frac{1}{2} \mathbf{i} \sigma^{\mu\nu} F_{\mu\nu} \epsilon + \mathbf{i}(|\phi|^2 - v)\epsilon$$

where the transformation parameter ϵ is a Majorana spinor (and a grassmann variable). Show that (something like this) is a symmetry of $\mathcal{L} = \mathcal{L}_h + \mathcal{L}_f$. This is $\mathcal{N} = 1$ supersymmetry in $D = 4$.

(b) Show that the conserved charges associated with these transformations Q_α (they are grassmann objects and spinors, since they generate the transformations, via $\delta_\epsilon$fields $= [\epsilon_\alpha Q_\alpha + h.c., \text{fields}]$), satisfy the algebra

$$\{Q, \bar{Q}\} = 2\gamma^\mu P_\mu + 2\gamma^\mu \Sigma_\mu \tag{10}$$

where P_μ is the usual generator of spacetime translations and Σ_μ is the *vortex string charge*, which is nonzero in a state with a vortex string stretching in the μ direction. $\bar{Q} \equiv Q^\dagger \gamma^0$ as usual.

(c) If we multiply (12) on the right by γ^0, we get the positive operator $\{Q_\alpha, Q^\dagger_\beta\}$. This operator annihilates states which satisfy $Q\,|BPS\rangle = 0$ for some components of Q. Such a state is therefore invariant under some subgroup of the superymmetry, and is called a BPS state. Now look at the right hand side of (12)$\times\gamma^0$ in a configuration where $\Sigma_3 = \pi n v^2$ and show that its energy density is $E \geq \pi |n| v^2$, with the inequality saturated only for BPS states.

(d) To find BPS configurations, we can simply set to zero the relevant supersymmetry variations of the fields. Since we are going to get rid of the fermion fields anyway, we can set them to zero and consider just the (bosonic) variations of the fermionic fields. Show that this reproduces the BPS equations.

This line of thought is the crucial ingredient by which progress has been made in understanding many supersymmetric theories, including superstring theories. For more, see for example here or chapter 14 of Polchinski's book.

3. **Wilson loops in abelian gauge theory at weak and strong coupling.**

(a) At weak coupling, the Wilson loop expectation value is a gaussian integral. In $D = 4$, study the continuum limit of a rectangular loop with time extent $T \gg R$, the spatial extent. Show that this reproduces the Coulomb force.

VI.B of this Kogut review explains this in some detail.

(b) Consider the weak coupling calculation again for a Wilson loop coupled to a massive vector field. Show that this reproduces an exponentially-decaying force between external static charges.

In this case the propagator is short-ranged, so as long as $R, T \gg m_A^{-1}$ the answer will be $\log \langle W(T,R)\rangle \simeq aR + bT$ a perimeter law.

(c) [bonus problem] Compute the combinatorial factors in the first few terms of the strong-coupling expansion of the Wilson loop in $\mathsf{U}(1)$ lattice gauge theory.

(d) [bonus problem] Consider the case of lattice gauge theory in two spacetime dimensions. In this case, show that the plaquette variables are actually independent variables.

In spacetime dimensions larger than two, any 3-volume V gives a relation between the plaquette variables, since

$$\prod_{\square \in \partial V} U_\square = 1.$$

This is because $U_\Box = \prod_{\ell\in\partial\Box} U_\Box$, and the plaquettes tiling the boundary of V (∂V) have boundaries which precisely cancel out so that the *boundary* of the boundary of V is empty. This is a general deep fact about topology: a boundary has no boundary. (This is the key ingredient in simplicial homology.)

But in $D = 2$, there are no 3-volumes, and hence no relations between the plaquette variables.

Here is another, related deep point: If we didn't realize that the boundaries of the plaquettes making up the boundary of V didn't cancel, we would write, in the abelian case,

$$\prod_{\Box\in\partial V} U_\Box = e^{\mathbf{i}e\sum_{\Box\in\partial_V}\oint_{\partial_\Box} A} \overset{\text{Stokes}}{=} e^{\mathbf{i}e\sum_{\Box\in\partial_V}\int_\Box F} = e^{\mathbf{i}e\int_{\partial_V} F} \overset{\text{Stokes}}{=} e^{\mathbf{i}e\int_V dF}.$$

But this last expression is the number of monopoles n inside the volume V times their charge g. But

$$e^{\mathbf{i}egn} = 1$$

is Dirac quantization.

In contrast, in two spacetime dimensions, there *are* no 3-volumes, so the plaquette variables are independent, and we can write the lattice gauge theory path integral, even for a non-abelian group, as

$$Z = \int \prod_\ell du_\ell e^{-S[U_\Box]} = \int \prod_\Box dU_\Box e^{-S[U_\Box]}$$

(perhaps up to some overall constant factor). For the special case where S is the Wilson action, the action is linear in the plaquette variables, so

$$Z = \int \prod_\Box dU_\Box e^{-\frac{1}{g^2}\sum_\Box \mathrm{tr} U_\Box} = \prod_\Box \left(\int dU_\Box e^{-\frac{1}{g^2}\mathrm{tr} U_\Box}\right) = \prod_\Box z_\Box = z_\Box^{\text{Area}}$$

where *Area* denotes the number of plaquettes. The theory just falls apart into independent plaquettes which don't care about each other. This extends to the evaluation of correlators of Wilson loops,

$$\langle W(C)\rangle = \prod_\Box \left(\frac{\int dU_\Box U_\Box e^{-\frac{1}{g^2}\mathrm{tr} U_\Box}}{z_\Box} \right) = w(\Box)^{\text{Area(C)}}$$

so the area law is exact. There are no propagating degrees of freedom, but the theory is not quite topological – it depends on areas. The object $z_\Box$ is a combination of characters of the group.

4. **Chern-Simons theory, flux attachment, and anyons.**

(a) Consider the following action for a $\mathsf{U}(1)$ gauge field in $D = 2 + 1$:

$$S[A] = \int \left(-\frac{1}{4g^2} F \wedge \star F + \frac{k}{4\pi} A \wedge F \right).$$

What are the dimensions of g and k? Which term (Maxwell or Chern-Simons) is more important for questions about low energy physics? Find the equations of motion for A. Look for plane wave solutions. Show that the resulting particle excitations have a mass which grows with g.

In components, this action is

$$S[A] = \int d^3x \left(-\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{k}{4\pi} \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} \right).$$

Since A is a gauge field, it has $[A] = 1$, so k is dimensionless and g^2 is a mass.

The Maxwell term is an irrelevant perturbation, while the CS term is marginal.

The equations of motion are

$$0 = \partial_\mu F^{\mu\lambda} + akg^2 \epsilon^{\lambda\nu\rho} F_{\nu\rho}$$

for some number $a = \frac{1}{\pi}$.

One nice way to see that this is a massive wave equation is to introduce the 'dual field strength' $F^\lambda \equiv \epsilon^{\lambda\rho\sigma} F_{\rho\sigma}$ (equivalently, $F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda} F^\lambda$). Then the EOM is

$$\epsilon^{\mu\nu\rho} \partial_\nu F_\rho + akg^2 F^\mu = 0.$$

Taking curl of both sides (*i.e.* acting with $\epsilon_{\mu\alpha\beta}\partial^\alpha$) and using $\epsilon^{\mu\nu\rho}\epsilon_{\mu\alpha\beta} = \delta^\nu_\alpha \delta^\mu_\beta - \delta^\mu_\alpha \delta^\nu_\beta$ gives

$$\left(\partial_\mu \partial^\mu + \left(akg^2\right)^2 \right) F_\rho = 0,$$

a massive wave equation for each component, with $M^2 = (akg^2)^2$ as the mass.

(b) For the rest of the problem, take $g \to \infty$. Notice that the resulting theory does not require a metric, since the action is made only from exterior derivatives and wedge products of forms. Now add a matter current j:

$$S_j[A] = \int \left(\frac{k}{4\pi} A \wedge F + A \wedge \star j \right).$$

Find the equations of motion. Show that the Chern-Simons term *attaches k units of flux* to the particles: $F_{12} \propto \rho$.

The last relation is the $\mu\nu = xy$ component of the EOM $\frac{k}{2\pi} F_{\mu\nu} + \epsilon_{\mu\nu\rho} j^\rho = 0$.

(c) Show using the Bohm-Aharonov effect that the particles whose current density is j^μ have anyonic statistics with exchange angle $\frac{\pi}{k}$ (supposing they were bosons before we coupled them to A).

One way to do this is to consider a configuration of j which describes one particle adiabatically encircling another. Show that its wavefunction acquires a phase $e^{\mathbf{i}2\pi/k}$. This is twice the phase obtained by going halfway around, which (when followed by an innocuous translation) would exchange the particles.