University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2022
Assignment 10

Due 11:59pm Monday, March 14, 2022

Thanks in advance for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

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1. **When is the QCD interaction attractive?**

   Write the amplitude for *tree-level* scattering of a quark and antiquark of different flavors (say u and $\bar{d}$) in the t-channel (in Feynman $\xi = 1$ gauge). Compare to the expression for $e\bar{\mu}$ scattering in QED.

   First fix the initial colors of the quarks to be different – say the incoming u is red and the incoming $\bar{d}$ is anti-green. Show that the potential is repulsive.

   Now fix the initial colors to be opposite – say the incoming u is red and the incoming $\bar{d}$ is anti-red – so that they may form a color singlet. Show that the potential is attractive.

   Alternatively or in addition, describe these results in a more gauge invariant way, by characterizing the potential in the color-singlet and color-octet channels.

   You can do this problem either by choosing a specific basis for the generators of $\mathsf{SU}(3)$ in the fundamental (a common one is called the Gell-Mann matrices), or using more abstract group theory methods.

2. **Where to find a Chern-Simons term.**

   Consider a field theory in $D = 2 + 1$ of a massive Dirac fermion, coupled to a background $\mathsf{U}(1)$ gauge field A with action:

   $$S[\psi, A] = \int d^3x \bar{\psi} \left(\mathbf{i} \not{D} - m \right) \psi$$

   where $D_\mu = \partial_\mu - \mathbf{i} A_\mu$.

   (a) Convince yourself that the mass term for the Dirac fermion in $D = 2 + 1$ breaks parity symmetry. By parity symmetry I mean a transformation $\psi(x) \to \Gamma \psi(Ox)$ where $\det O = -1$, and Γ is a matrix acting on the spin indices, chosen so that this operation preserves $\bar{\psi} \not{\partial} \psi$.

(b) We would like to study the effective action for the gauge field that results from integrating out the fermion field

$$e^{-S_{eff}[A]} = \int [D\psi D\bar{\psi}] e^{-S[\psi,A]}.$$

Focus on the term quadratic in A:

$$S_{eff}[A] = \int d^D q A_\mu(q) \Pi^{\mu\nu}(q) A_\nu(q) + ...$$

We can compute $\Pi^{\mu\nu}$ by Feynman diagrams[1]. Convince yourself that Π comes from a single loop of ψ with two A insertions.

(c) Evaluate this diagram using dim reg near $D = 3$. Show that, in the low-energy limit $q \ll m$ (where we can't make on-shell fermions),

$$\Pi^{\mu\nu} = a \frac{m}{|m|} \epsilon^{\mu\nu\rho} q_\rho + ...$$

for some constant a. Find a. Convince yourself that in position space this is a Chern-Simons term with level $k = \frac{1}{2}\frac{m}{|m|}$.
[Hint: in $D = 2+1$, $\text{tr}\gamma^\mu\gamma^\nu\gamma^\rho = -2\epsilon^{\mu\nu\rho}$.]

(d) Redo this calculation by doing the Gaussian path integral over ψ.

3. **A bit more about Chern-Simons theory.**

Consider again $\mathsf{U}(1)$ gauge theory in $D = 2+1$ dimensions with the Chern-Simons action

$$S[a] = \frac{k}{4\pi} \int_\Sigma a \wedge da.$$

(Here I've changed the name of the dynamical gauge field to a lowercase a to distinguish it from the electromagnetic field A which will appear anon.)

(a) Show that the Chern-Simons action is gauge invariant under $a \to a + d\lambda$, as long as there is no boundary of spacetime Σ. Compute the variation of the action in the presence of a boundary of Σ.

(b) [bonus] Actually, the situation is a bit more subtle than the previous part suggests. The actual gauge transformation is

$$a \to g^{-1} a g + \frac{1}{\mathbf{i}} g^{-1} dg$$

which reduces to the previous if we set $g = e^{\mathbf{i}\lambda}$. That expression, however, ignores the global structure of the gauge group (*e.g.* in the abelian case,

[1] The thing I've called $\Pi^{\mu\nu}$ here is actually twice the vacuum polarization. Sorry.

the fact that g is a periodic function). Consider the case where spacetime is $\Sigma = S^1 \times S^2$, and consider a *large gauge transformation*:

$$g = e^{\mathbf{i}n\theta}$$

where θ is the coordinate on the circle. Show that the variation of the CS term is $-\mathbf{i}\frac{k}{4\pi}\int g^{-1}dg \wedge f$ (where $f = da$). Since the action appears in the path integral in the form $e^{\mathbf{i}S}$, convince yourself that the path integrand is gauge invariant if

(1) $\int_\Gamma f \in 2\pi\mathbb{Z}$ for all closed 2-surfaces Γ in spacetime, and

(2) $k \in 2\mathbb{Z}$ – the Chern-Simons level is quantized as an *even* integer.

The first condition is called flux quantization, and is closely related to Dirac's condition.

The quantization of the level k, i.e. the Chern-Simons coupling has a dramatic consequence: it means that this coupling constant cannot be renormalized by a little bit, only by an integer shift. This is an enormous constraint on the dynamics of the theory.

(c) [bonus] In the case where G is a non-abelian lie group, the argument for quantization of the level k is more straightforward. Show that the variation of the CS Lagrangian

$$\mathcal{L}_{CS} = \frac{k}{4\pi}\mathrm{tr}\left(a \wedge da + \frac{2}{3}a \wedge a \wedge a\right)$$

under $a \to gag^{-1} - dgg^{-1}$ is

$$\mathcal{L}_{CS} \to \mathcal{L}_{CS} + \frac{k}{4\pi}d\mathrm{tr} dgg^{-1} \wedge a + \frac{k}{12\pi}\mathrm{tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right).$$

The integral of the second term over any closed surface is an integer. Conclude that $e^{\mathbf{i}S_{CS}}$ is gauge invariant if $k \in \mathbb{Z}$.

(d) Now we return to the abelian case (for an extra challenge, redo this part in the non-Abelian case). If there is a boundary of spacetime, something must be done to fix up this problem. Consider the case where $\Sigma = \mathbb{R} \times \mathrm{UHP}$ where $\mathbb{R}$ is the time direction, and UHP is the upper half-plane $y > 0$. One way to fix the problem is simply to declare that the would-be gauge transformations which do not vanish at $y = 0$ are not redundancies. This means that they represent physical degrees of freedom. Plug in $a = d\phi$ to the Chern-Simons action (where $\phi(x, y \to 0) \equiv \phi(x)$ is a scalar field, and d is the exterior derivative on the spatial manifold) to find the action for ϕ.

We can also add local terms at the boundary to the action. Consider adding $\Delta S = g \int_{\partial\Sigma} a_x^2$ (for some coupling constant g). Find the equations of motion for ϕ.

Interpretation: the Chern-Simons theory on a space with boundary necessarily produces a chiral edge mode.

(e) Suppose we had a system in $2+1$ dimensions with a gap to all excitations, which breaks parity symmetry and time-reversal invariance, and involves a conserved current J^μ, with

$$0 = \partial^\mu J_\mu. \tag{1}$$

Solve this equation by writing $J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu a_\rho$ in terms of a one-form $a = a_\mu dx^\mu$. Guess the leading terms in the action for a_μ in a derivative expansion. You may assume Lorentz invariance.

(f) Now suppose the current J^μ is coupled to an external electromagnetic field A_μ by $S \ni \int J^\mu A_\mu$. Ignoring the Maxwell term for a, compute the Hall conductivity, σ^{xy}, which is defined by Ohm's law $J^x = \sigma^{xy} E^y$.