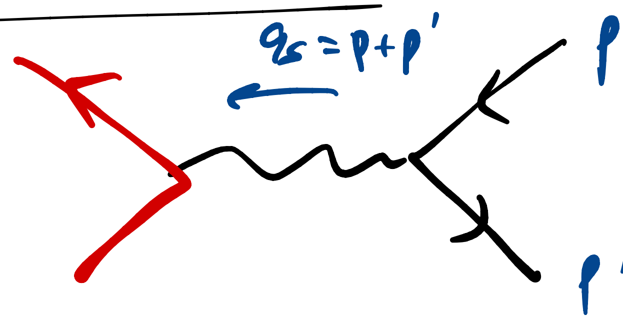
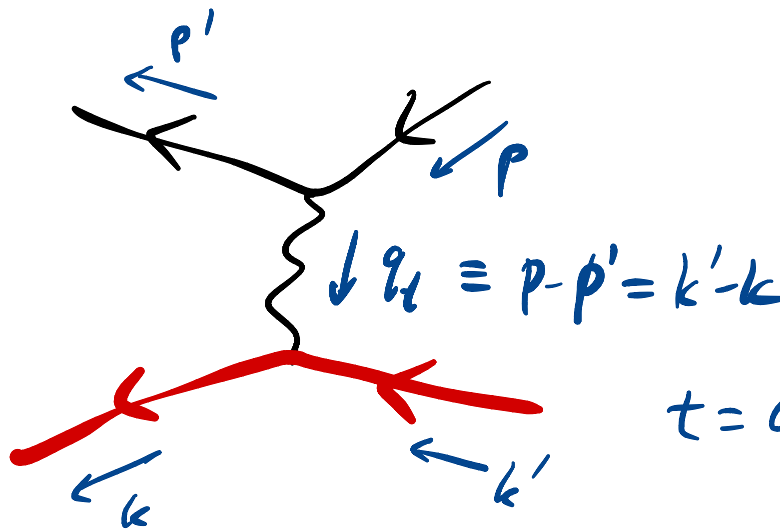
1.3 Towards Quantum Corrections to the Coulomb Force Law

$\boxed{e^+e^- \to \mu^+\mu^-}$



$\boxed{e^-\mu^- \leftarrow e^-\mu^-}$

$i\mathcal{M} =$



$t = q_t^2.$

$$= \left(-ie\,\bar{u}(p')\gamma^\mu \underline{\underline{u(p)}}\right)_{el.} \frac{-i\left(\eta_{\mu\nu} - \dots\right)}{q_t^2} \times$$

$\not{p}u = m_e u$

$$\left(-ie\;\bar{u}(k)\gamma^\nu u(k')\right)_{muons}$$

$\not{k}u = m_\mu u$

$$\frac{1}{4}\sum_{ss'rr'} |\mathcal{M}|^2 = \frac{1}{4}\frac{e^4}{t^2} E^{\mu\nu} M_{\mu\nu}$$

$$= \frac{1}{4}\frac{e^4}{t^2}\left(-p^\mu p'^\nu - p'^\mu p^\nu - \eta^{\mu\nu}(-p\cdot p' + m_e^2)\right)$$

$$\left(-k_\mu k'_\nu - k'_\mu k_\nu - \eta_{\mu\nu}(-k\cdot k' + m_\mu^2)\right)$$

[Relative to $e^+e^- \to \mu^+\mu^-$:

$(s, t, u) \longrightarrow (t, u, s)$.]

Kinematics : 'Heavy' charge :

CoM frame ~ its rest frame.

$$k_0' = m_\mu,\quad k_0 = \sqrt{m_\mu^2 + \vec{k}^2}$$

$$= m_\mu + \frac{\vec{k}^2}{2m_\mu} + \ldots$$

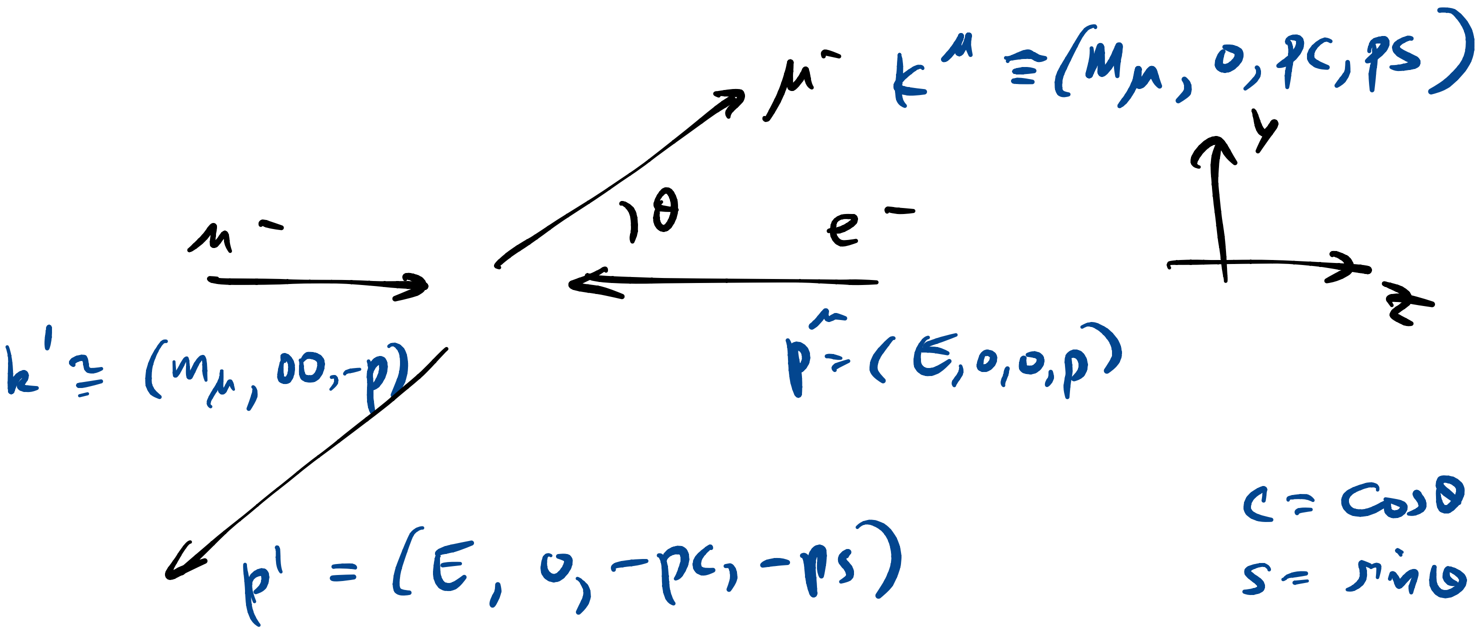
(elastic) $\Longleftarrow$ $\simeq m_\mu$.

Same as : fixed static potential $A_0 = \frac{Ze}{r}$

Z = charge of heavy charge

Note: one feynman diagram (no sums)

$\Rightarrow$ classical.



$$-\frac{1}{4} M_{\mu\nu} = k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu} \big(\underbrace{k\cdot k' - m_\mu^2}_{\doteq\, m_\mu^2 - m_\mu^2 = 0}\big)$$

$$\doteq \delta_{\mu,0}\, \delta_{\nu 0}\, 2 m_\mu^2 .$$

$$-p\cdot p' + m_e^2 = -E^2 + \vec{p}^{\,2} \cos\theta + m_e^2$$

$$= -\vec{p}^{\,2}(1 - \cos\theta)$$

$$\overset{\text{trig}}{=} -\vec{p}^{\,2}\, 2 \sin^2 \theta/2$$

$$\epsilon^{\mu\nu} M_{\mu\nu} = 32 m_\mu^2 \left(2E^2 + \eta^{00}\,(p \cdot p' - m_e^2)\right)$$

$$= 2 \cdot 32\, m_\mu^2 \left(E^2 - \vec{p}^{\,2} \sin^2\frac{\theta}{2}\right)$$

$$\overset{\beta^2 = \vec{p}^{\,2}/E^2}{=} 64\, m_\mu^2 E^2 \left(1 - \beta^2 \sin^2\frac{\theta}{2}\right).$$

$$d\sigma_{\substack{2\to 2\\ \text{com}}} = \frac{1}{v_{rel}} \underline{\underline{\frac{1}{2E_A}}}\ \underline{\underline{\frac{1}{2E_B}}} \underbrace{\left(\frac{1}{4}\sum |M|^2\right)} d\Pi_{LI}$$

$$= \frac{1}{\beta} \frac{1}{2E} \frac{1}{2m_\mu} \frac{z^2 e^4}{t^2} \frac{64}{4} m_\mu^2 E^2 \left(1 - \beta^2 \sin^2\frac{\theta}{2}\right)$$

$$\frac{d\Omega}{16\pi^2} \frac{p}{E_{tot}}.$$

$$t = (p-p')^2 = -2\vec{p}^{\,2}(1-\cos\theta)$$

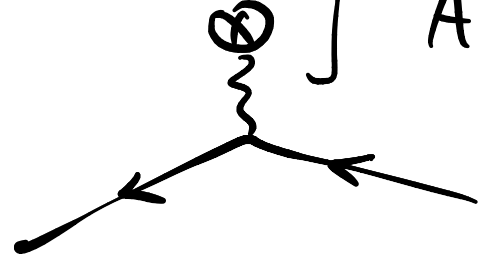
$$\overset{E_{tot} \sim m_\mu}{=} \frac{4Ep}{\beta} z^2 \alpha^2 \frac{1 - \beta^2 \sin^2\frac{\theta}{2}}{t^2}\, d\Omega.$$

$$\alpha = \frac{e^2}{4\pi}$$

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{Mott}} = z^2 \alpha^2 \frac{1 - \beta^2 \sin^2\frac{\theta}{2}}{4\beta^2 \vec{p}^{\,2} \sin^4\theta/2}.$$

Comments about Mott formula:

- Indep of m_μ, might as well be ∞.

  same as  $A_0 = \frac{Ze}{r}$.

- $\beta \ll 1$ $\rightsquigarrow$ Rutherford formula.

- $\theta \to 0$ $\rightsquigarrow$ ∞. (photon is massless)

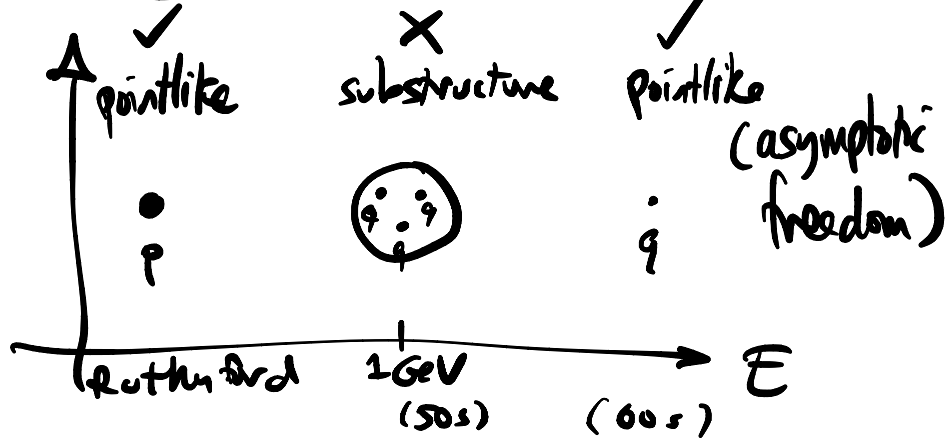
- for $E \gg m_e$ must account for

  recoil of m_μ.

to get a **general formula** for

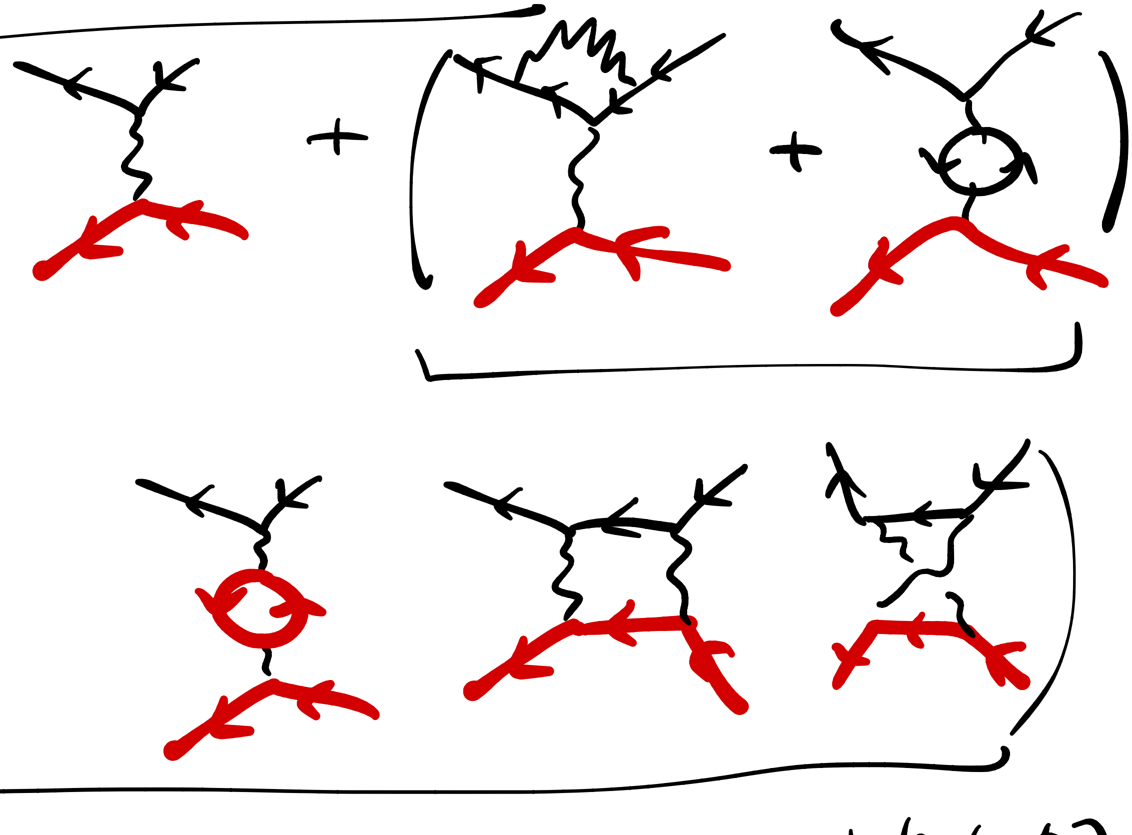
pointlike classical Coulomb scattering

eg: scatter e^- off a p at various E.



Radiative corrections:

$i\mathcal{M}_{e\mu\leftarrow e\mu} =$



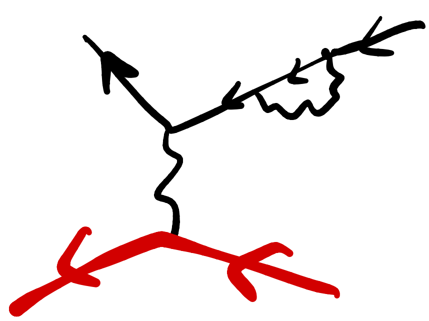
$+\,\mathcal{O}(e^6)$

- Suppose we're interested in $E \ll m_\mu$.

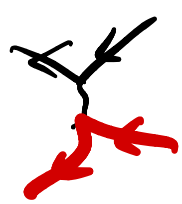
$$\frac{i}{\not k - m_\mu} \overset{k\ll m_\mu}{\sim} \frac{1}{m_\mu}$$

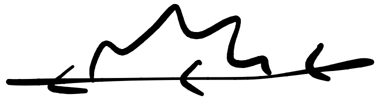
suppressed by $\frac{k}{m_\mu} \ll 1$.

- what about:



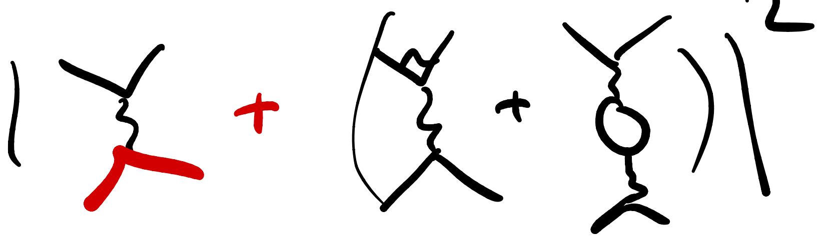
?

$S_{e\mu\leftarrow e\mu} = \sqrt{Z_e}^2 \sqrt{Z_\mu}^2 \Big($  $+ \dots \Big)_{\text{amputated on-shell}}$

i.e.  is a correction to the propagator $\rightsquigarrow$ Z_e and δm_e.

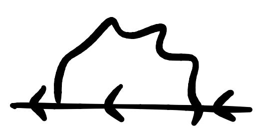
- $\mathcal{M} = \mathcal{M}_{tree} + O(\alpha^2)$ $\qquad \alpha = e^2/4\pi$

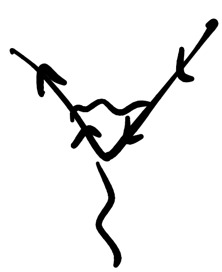
$\uparrow$
$O(\alpha)$

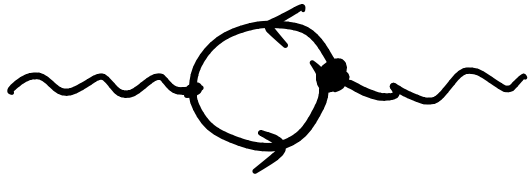
$\sigma \sim$ 

$\sim \sigma_{tree} + O(\alpha^3)$

3 'primitive' UV divergent processes in QED.

- e^- self energy  emission & absorption of γs obstructs the propagation of e^-.

- vertex correction 

- vacuum polarization 

(photon self-energy)

Dielectric medium

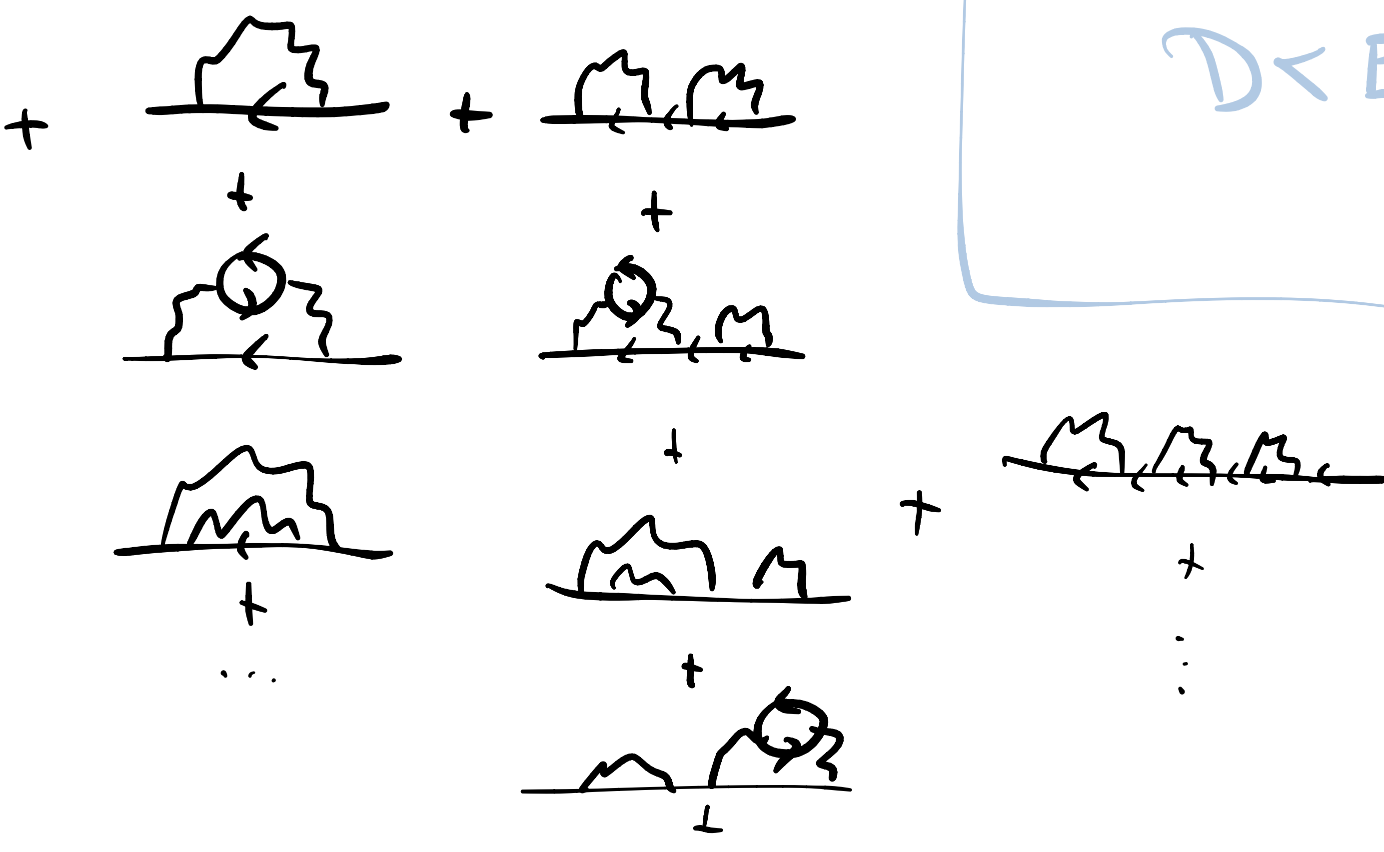
$E = 0$

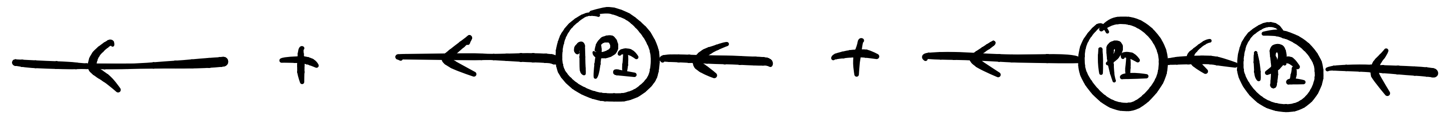
$E_{applied}$

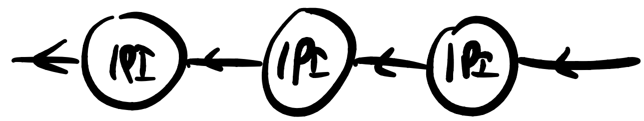
$D < E_{applied}$

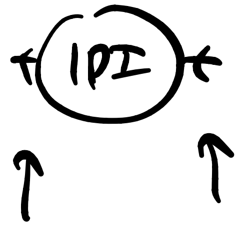
1.4 Electron self-energy in QED

$\tilde{G}^{(2)}(p) =$ 



$=$ 

$+$ 

 $\equiv -i\Sigma(p) \equiv$ sum of all 1PI diagrams w/ 1 nubbin in, 1 nubbin out.

$\equiv$ self-energy

$\equiv$ can't be broken in 2 by cutting any one propagator.

$$iS(p) \equiv \frac{i}{\not{p} - m_0} =$$ 

$$= \frac{i(\not{p} + m_0)}{p^2 - m_0^2 + i\epsilon}$$

↑ mass in $\mathcal{L}$

<u>Note</u>: $\tilde{G}, S, \Sigma$ are matrices in spinor space.

$$\tilde{G}(p) = iS(\not{p}) + iS(\not{p})(-i\Sigma(\not{p}))\,iS(\not{p}) + iS\,(-i\Sigma)\,iS\,(-i\Sigma)\,iS + \dots$$

$$= iS\left(1 + \Sigma S + \Sigma S \Sigma S + \dots\right)$$

$$= iS\,\frac{1}{1-\Sigma S}$$

$$= \frac{i}{\not{p}-m_0}\,\frac{1}{1-\Sigma\frac{1}{\not{p}-m_0}}$$

$$= \frac{i}{\not{p}-m_0-\Sigma(\not{p})}$$

note: $S = S(\not{p})$

$\Sigma = \Sigma(\not{p})$

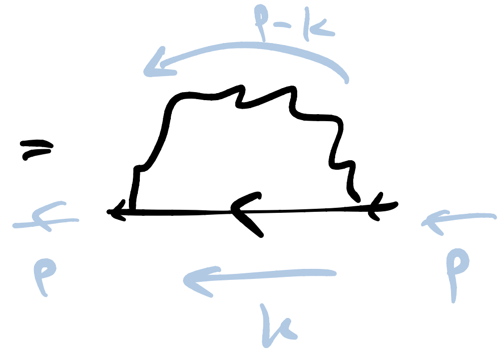
$p^2 = (\not{p})^2$.

Do this in eigenbasis of $\not{p}$

The corrected propagator has a pole at

$$\not{p} = \underbrace{m\mathbb{1} = m_0\mathbb{1} + \Sigma(m\mathbb{1})}_{\text{defines } m \,\equiv\, \text{Renormalized mass.}}\ .$$

Leading Contribution to $\Sigma = \Sigma_2 + O(\alpha^2)$

$-i\Sigma_2(p) =$ 

p is arbitrary

$$= (-ie)^2 \int d^4k \ \gamma^\mu \ \underline{\underline{iS(k)}} \gamma^\nu \ \frac{-i\eta_{\mu\nu}}{(p-k)^2 - \mu^2 + i\epsilon}$$

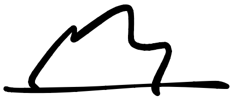
mass for γ

Feynman Integrals:

step 1: Feynman parameter trick $\frac{1}{AB} = \int_0^1 dx \frac{1}{C^2}$

$$\int_0^1 dx \frac{1}{(xA + (1-x)B)^2} = \int_0^1 \frac{dx}{(x(A-B)+B)^2}$$

$$= \frac{1}{A-B} \frac{-1}{x(A-B)+B} \Big|_{x=0}^{x=1} = \frac{1}{A-B}\left(-\frac{1}{A} + \frac{1}{B}\right) = \frac{1}{AB}$$

 $= -e^2 \int d^4k \, N \, \mathcal{D}$

$$N \equiv \gamma^\mu (\not{k} + m_0) \gamma_\mu$$

$$\mathcal{D} \equiv \underbrace{\frac{1}{k^2 - m_0^2 + i\epsilon}}_{B} \; \underbrace{\frac{1}{(p-k)^2 - \mu^2 + i\epsilon}}_{A}$$

$$\overset{\text{Feynman}}{\underset{\text{trick}}{=}} \int_0^1 dx \frac{1}{\Big(\underbrace{x\big((p^2 - 2p\cdot k + k^2) - \mu^2 + i\epsilon\big) + (1-x)(k^2 - m_0^2 + i\epsilon)}_{= k^2 - 2x\, k\cdot p + (0) k^0}\Big)^2}$$

$x + (1-x) = 1$

$$= (k - xp)^2 - \#$$

Step 2 : Complete the square

$$\mathcal{D} = \int_0^1 dx \left(\frac{1}{\underbrace{(k - px)}_{\equiv \ell}{}^2 - \Delta + i\epsilon} \right)^2$$

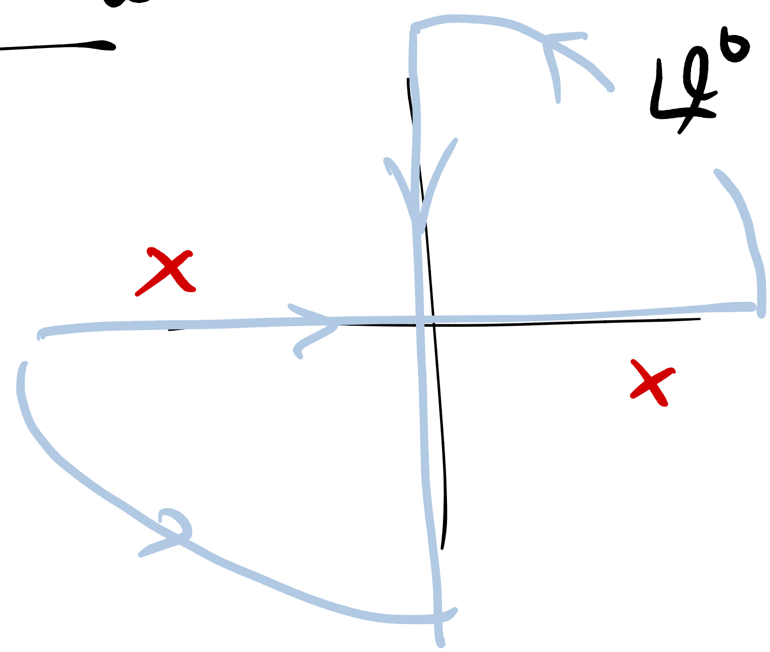
$\rightsquigarrow \ell^\mu = k^\mu - p^\mu x$

$$\Delta(\mu^2) \equiv p^2 x^2 + x\mu^2 - x p^2 + (1-x) m_0^2$$

$$= x\mu^2 + (1-x) m_0^2 - x(1-x) p^2 .$$

Step 3: Wick Rotate.

$l^0 = k^0 - p^0 x$



$$i\epsilon \Longrightarrow \int_{\mathbb{R}} dl^0 \ldots$$

$$= \int_{\text{Im. axis}} dl_E^4 \ldots$$

$l^0 \equiv i l_E^4$

$$l^2 = l^\mu l_\mu = -l_E^2 = -\left(\sum_{i=1}^{4} l_{E_i}^2\right)$$

$$-i\Sigma_2(p) = -e^2 \int d^4 l \int_0^1 dx \frac{N}{(l^2 - \Delta + i\epsilon)^2}$$

$$= -e^2 \int_0^1 dx \; i \int_{R^4} d^4 l_E \frac{N}{(l_E^2 + \Delta)^2} .$$

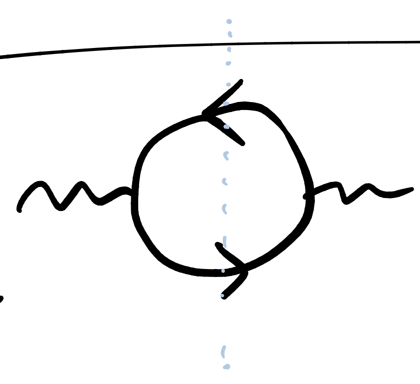
$$\int d^4 l_E \frac{1}{(l_E^2+\Delta)^2} \sim \int^{\Lambda} \frac{d^4 l}{l^4} \sim \log \Lambda .$$

$\exists \Lambda$ s.t.

$E \to \Lambda^n E \qquad n \in \mathbb{Z}$

preserves the spectrum.

$$\{ E_n = E_0 \Lambda^n \}$$

Im () $\propto \sum_f$ |  $|^2$