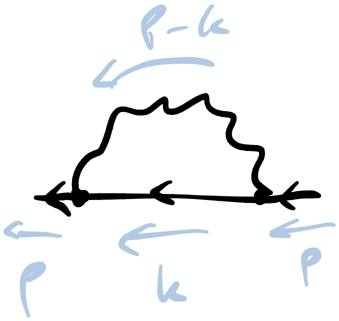
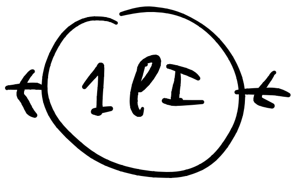
1.4 Electron Self-Energy, continued



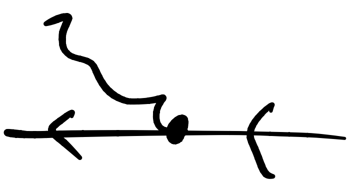
$$= -i\Sigma_2(\not p) = -e^2 \int d^4k \, N \mathcal{D}$$

$$N = \gamma^\mu (\not k + m_0) \gamma_\mu$$

$$\mathcal{D} = \frac{1}{AB} = \frac{1}{k^2 - m_0 + i\epsilon} \frac{1}{(p-k)^2 - \mu^2 + i\epsilon}$$



$$= \int_0^1 dx \frac{1}{\left(k^2 - 2x k\cdot p + \odot\right)^2}$$



$$= \int_0^1 dx \frac{1}{\left(\ell^2 - \Delta + i\epsilon\right)^2}$$

$$\ell^\mu \equiv k^\mu - p^\mu x .$$

$$\Delta(\mu) = x\mu^2 + (1-x)m_0^2 - x(1-x)p^2 .$$

$$-i\Sigma_2(\not p) = -e^2 \int_0^1 dx \, i \int d^4\ell_E \frac{N}{\left(\ell_E^2 + \Delta\right)^2}$$

$$N = \gamma^\mu (\not{l} + x \not{p} + m_0) \gamma_\mu$$

$$\overset{\text{Clifford}}{=} -2 (\not{l} + x \not{p}) + 4 m_0$$

$$\int d^4 l_E \frac{l_E^\mu}{(l_E^2 + \Delta)^2} \overset{\text{Rot. inv.}}{=} 0$$

$$d^4 l_E = \frac{1}{(2\pi)^4} d\Omega_3 \underbrace{l^3 dl}_{= \frac{l^2 dl^2}{2}}$$

$$\rightarrow \Sigma_2(\not{p}) = e^2 \int_0^1 dx \int \frac{l^2 dl^2}{2} \frac{(2\pi^2)}{(2\pi)^4} \frac{2 (2m_0 - x\not{p})}{(l^2 + \Delta^2)^2}$$

$$= \frac{e^2}{8\pi^2} \int_0^1 dx \, (2m_0 - x\not{p}) J_\Delta$$

$$J_\Delta = \int_0^\infty dl^2 \frac{l^2}{(l^2 + \Delta)^2} .$$

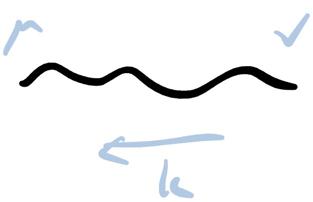
$$J_\Delta = \int dl^2 \left(\underbrace{\frac{l^2+\Delta}{(l^2+\Delta)^2}} - \frac{\Delta}{(l^2+\Delta)^2} \right)$$

$$= \ln(l^2+\Delta)\Big|_{l^2=0}^{\infty} + \frac{\Delta}{l^2+\Delta}\Big|_{l^2=0}^{\infty}$$

$$= \underbrace{\ln(l^2+\Delta)\Big|_{l^2=0}^{\infty}} - 1$$

Pauli-Villars Regulator

$$\boxed{H_{charge} = \frac{(\widetilde{p+A})^2}{2m}}$$



$$\rightsquigarrow -i\eta_{\mu\nu}\left(\frac{1}{k^2-\mu^2+i\epsilon} - \frac{1}{k^2-\Lambda^2+i\epsilon} \right)$$

$$= -i\eta_{\mu\nu}\left(\frac{\mu^2-\Lambda^2}{(k^2-\mu^2+i\epsilon)(k^2-\Lambda^2+i\epsilon)} \right)$$

$$\overset{k^2 \gg \Lambda^2}{\sim} \frac{1}{k^4}.$$

$$\Rightarrow \int \frac{d^4k}{k^2k^2} \rightsquigarrow \int \frac{d^4k}{k^2k^4} \text{ finite.}$$

$$D^{PV}_{\mu\nu}(k) \overset{k^2 \ll \Lambda^2}{\sim} D_{\mu\nu}(k) + \frac{-i\eta_{\mu\nu}}{\Lambda^2}$$

$$\longrightarrow \frac{-i\eta_{\mu\nu}}{k^2-\mu^2+i\epsilon}$$

The PV photon is a **ghost**

($\underline{Z < 0}$.) unitary?

<u>ok?</u> $\Lambda \gg$ everything

$\Rightarrow$ never make a PV photon.

$$A_\mu \to A_\mu + \partial_\mu \underline{\underline{\lambda}}$$

<u>virtues of PV</u>:
- Lorentz inv't
- gauge inv't ✓
- easy.

<u>shortcoming</u>:
- not unitary !
- not non-perturbat-ively well-defined.

	gauge inv	Lorentz inv	non-perturbative	unitary	easy
PV	✓	✓	✗	✗	✓
hard cutoff	✗	✓	~	✓	✓
lattice	✓	✗	✓	✓	✗
dim reg	✓	✓	?	?	✓
⋮					

$$\mathcal{J}_\Delta \rightsquigarrow \mathcal{J}_{\Delta(\mu)} - \mathcal{J}_{\Delta(\Lambda)}$$

$$= \left[\ln\left(l^2+\Delta(\mu)\right) - 1 - \left(\ln\left(l^2+\Delta(\Lambda)\right)-1\right)\right|_{l=0}^{\infty}$$

$$= \ln \frac{l^2+\Delta(\mu)}{l^2+\Delta(\Lambda)}\Bigg|_{l=0}^{\infty}$$

$$= \ln 1/1 - \ln\frac{\Delta(\mu)}{\Delta(\Lambda)} = \ln\frac{\Delta(\Lambda)}{\Delta(\mu)}.$$

$$\Delta(\Lambda) = x\Lambda^2 + (1-x)m_0^2 - x(1-x)p^2 \overset{\Lambda \gg \dots}{\sim} x\Lambda^2 .$$

$$\Sigma_2(\not{p})\Big|_{PV} = \frac{\alpha}{2\pi}\int_0^1 dx\,(2m_0 - x\not{p})\,\times$$

$$\ln\frac{x\,\Lambda^2}{x\mu^2 + (1-x)m_0^2 - x(1-x)p^2}\,.$$

"Impose a renormalization condition":

$\tilde{G}(\not{p})$ has a pole at $\not{p} = m = m_0 + \Sigma(m)$

$$\delta m \equiv m - m_0 = \Sigma_2(\not{p} = m) + O(e^4)$$

$$= \Sigma_2(\not{p} = m_0) + O(e^4)$$

$$= \frac{\alpha}{2\pi}\int_0^1 dx\,(2-x)\,m_0 \ln\frac{x\,\Lambda^2}{f(x, m_0, \mu)}$$

$$f(x, m_0, \mu) \equiv x\mu^2 + (1-x)^2 m_0^2\,.$$

$$= \frac{\alpha}{2\pi}\int_0^1 dx\,(2-x)\,m_0\left(\underbrace{\ln\frac{\Lambda^2}{m_0^2}}_{\text{divergent}} + \underbrace{\ln\frac{m_0^2\,x}{f(x, m_0, \mu)}}_{\text{finite (relatively small.)}}\right)$$

$$\delta m \cong \frac{\alpha}{2\pi}\left(2-\frac{1}{2}\right) m_0 \ln \frac{\Lambda^2}{m_0^2}$$

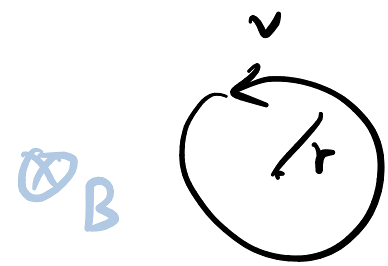
$$= \frac{3\alpha}{4\pi} m_0 \ln \frac{\Lambda^2}{m_0^2}.$$

Mass Renormalization:

fiction

$$511\,\text{keV} \simeq m_e \overset{!}{=} m_0\left(1+\frac{3\alpha}{4\pi}\ln\frac{\Lambda^2}{m_0^2} + O(\alpha^2)\right)$$

measure.



$$m_e \frac{v^2}{r} = \frac{evB}{c}$$

solve for $m_0(\Lambda)$ to $O(\alpha^2)$

Wavefunction Renormalization:

$$\tilde{G}^{(2)}(p) = \frac{i}{\not{p} - m_0 - \Sigma(p)} \underset{p\sim m}{\simeq} \frac{iZ}{\not{p}-m}$$ + regular terms at $\not{p}=m$

$$\Sigma(\cancel{p}) \overset{\text{Taylor}}{=} \Sigma(\cancel{p}=m) + \left.\frac{\partial \Sigma}{\partial \cancel{p}}\right|_{\cancel{p}=m} (\cancel{p}-m) + \mathcal{O}(\cancel{p}-m)^2$$

$$= \Sigma_2(\cancel{p}=m_0) + \left.\frac{\partial \Sigma_2}{\partial \cancel{p}}\right|_{\cancel{p}=m_0} (\cancel{p}-m_0) + \mathcal{O}(\cancel{p}-m_0)^2 + \mathcal{O}(e^4)$$

$$\Rightarrow \tilde{G}^{(2)}(p) \overset{\cancel{p}\sim m}{\sim}$$

$$\frac{i}{\underline{\cancel{p}-m_0} - \underline{\left.\frac{\partial \Sigma}{\partial \cancel{p}}\right|_{m_0}} (\cancel{p}-m) + \mathcal{O}(\cancel{p}-m_0)^2 + \mathcal{O}(e^4)}$$

$m_0 - \Sigma_2(\cancel{p}-m_0)$
$= m + \mathcal{O}(\alpha^2)$

$$= \frac{i}{(\cancel{p}-m)\left(1 - \left.\frac{\partial \Sigma}{\partial \cancel{p}}\right|_{m_0}\right)} + \mathcal{O}(\cancel{p}-m)^0 + \mathcal{O}(\alpha^2).$$

regular at $\cancel{p}=m$

$$\frac{1}{\underline{\underline{x-a}} + \underline{\underline{(x-a)^2}}} = \frac{1}{\underline{\underline{x-a}}} \Bigg(\underbrace{\frac{1}{1+(x-a)}}_{1-(x-a)+\dots} \Bigg)$$

$$= \frac{1}{x-a} - \underset{\substack{\uparrow \\ (x-a)^0}}{1} + \dots$$

$$\underline{\underline{Z = \frac{1}{1 - \left.\frac{\partial \Sigma}{\partial \not p}\right|_{m_0}}}} \simeq \underline{\underline{1 + \left.\frac{\partial \Sigma_2}{\partial \not p}\right|_{m_0}}} + \underline{\underline{\mathcal{O}(\alpha^2)}}$$

$$\delta Z = \left.\frac{\partial \Sigma_2}{\partial \not p}\right|_{m_0} = \frac{\alpha}{2\pi} \int_0^1 dx \Bigg(-\underline{x} \ln \frac{x\Lambda^2}{f(x, m_0, \mu)}$$

$$+ (2m_0 - x m_0) \frac{-2x(1-x)}{f(x, m_0, \mu)} \Bigg]$$

$$= -\frac{\alpha}{4\pi} \left(\ln \frac{\Lambda^2}{m_0^2} + \text{finite} \right).$$

appears in the S-matrix !!?

$$Z \sim \left| \langle p | \psi(x) | 0 \rangle \right|^2$$

1-particle state.

$$\delta m = \Sigma_2 \Big|_{m_0}$$

$$\delta Z = \frac{\partial \Sigma_2}{\partial p} \Big|_{m_0}.$$

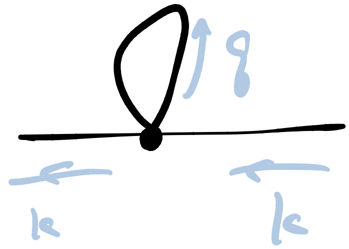
1.5 Big picture Interlude:

Self-energy in ϕ^4 theory

$$\mathcal{L} = -\frac{1}{2}\left(\underline{\phi \Box \phi} + m^2 \phi^2\right) - \frac{g_P}{4!}\phi^4 + \mathcal{L}_{ct}$$

physical m, g

$$\mathcal{L}_{ct} = -\frac{1}{2}\underline{\delta Z\, \phi \Box \phi} - \frac{1}{2}\delta_{m^2}\phi^2 - \underline{\frac{\delta_g}{4!}}\phi^4.$$

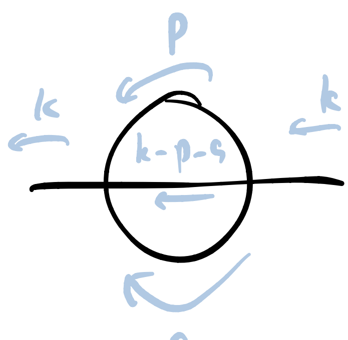
$$\delta\Sigma_1(k) =$$ 

$$= -ig \int^{\Lambda} d^4q \, \frac{i}{q^2 - m^2 + i\epsilon}$$ indep. of k!

$$= \delta\Sigma_1(k=0) \sim g\Lambda^2.$$

Demanding the pole in $G(p)$ is at $p^2 = m^2$

$$\Rightarrow \delta_{m^2} = -\delta\Sigma_1.$$
$$\delta_Z = 0 + O(g^2)$$

$$\delta\Sigma_2(k) =$$ 

$$= \frac{(-ig)^2}{3!} \int^{\Lambda} d^4p \int^{\Lambda} d^4q \; iD_0(p)\, iD_0(q)\, iD_0(k-p-q)$$

$$\equiv I(k^2, m, \Lambda)$$

$$\sim \int^{\Lambda} \frac{d^8p}{p^6} \sim \Lambda^2.$$

$$\delta\Sigma_2(k^2) = \underline{A_0} + k^2 \underline{A_1} + k^4 \underline{A_2} + \dots$$

$k^2 \gg \dots$

$\boxed{\text{to see UV behavior}}$

$$A_n = \frac{1}{n!}\left(\frac{\partial}{\partial k^2}\right)^n \delta\Sigma_2 \Big|_{k^2=0}$$

$$A_0 = I(k^2=0) \sim \Lambda^2 .$$

$$A_1 = \frac{\partial}{\partial k^2} I \Big|_{k^2=0} \sim \int^{\Lambda} \frac{d^8P}{P^8} \sim \ln\Lambda$$

$$A_2 = \frac{1}{2}\left(\frac{\partial}{\partial k^2}\right)^2 I \Big|_{k^2=0} \sim \int^{\Lambda} \frac{d^8P}{P^{10}} \sim \Lambda^{-2}$$

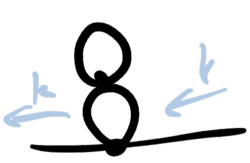
$\rightarrow$ finite as $\underline{\Lambda \to 0.}$

$\vdots$

at $\underline{k^2 = m_p^2}$: $\quad A_n = \frac{1}{n!}\left(\frac{\partial}{\partial k^2}\right)^n \delta\Sigma_2 \Big|_{k^2=m_p^2}$.

$$D^{-1}(k) = D_0^{-1}(k) - \Sigma(k^2) = k^2 \underbrace{- m_0^2 - \left(\delta\Sigma_1(m_p^2) + A_0 + B_0\right)}_{= -m_p^2}$$

$$- (k^2 - m_p^2) A_1 - (k^2 - m_p^2)^2 A_2 + \dots$$



$= B_0$

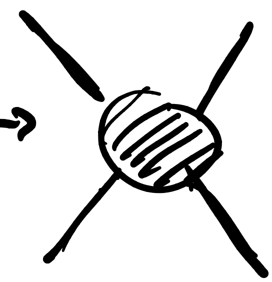
$$D(k) \overset{k^2\sim m_p^2}{\cong} \frac{1}{(1-A_1)(k^2-m_p^2)} + (k^2-m_p^2)^0 + \dots$$

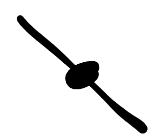
$$= \frac{Z}{k^2-m_p^2} + \text{regular}$$

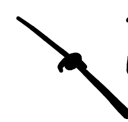
- $Z = \underline{\underline{\frac{1}{1-A_1}}} + O(g^3) = 1 + A_1 + \underline{\underline{O(g^3)}}$

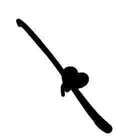
- If $A_{n\geq 2}$ had been cutoff-dependent we would need a counterterm

$$\delta_n \, \phi \, \Box^n \phi$$

$\phi^4 \rightarrow$ 

$\phi^2 \rightarrow$ 

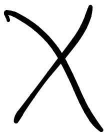
$\phi \Box \phi \rightarrow$  k^2

$\phi \Box^2 \phi \rightarrow$  k^4.

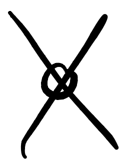
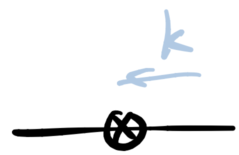
- $\delta m^2 \sim \Lambda^2$ vs $\delta m_e \sim \ln \Lambda$ in QED

Rules for Renormalized pert thy:

- add a c.t. for every term in $\mathcal{L}$.

 $= -ig_P$ $= \dfrac{i}{k^2 - m_P^2 + i\epsilon}$

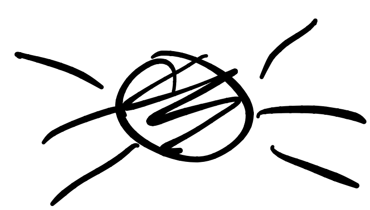
choose the counterterms to make it so.

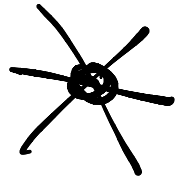
 $= -i\delta_g$  $= -i\left(\delta_Z k^2 + \delta m^2\right)$

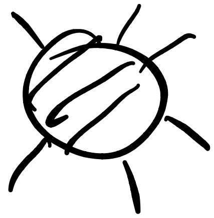
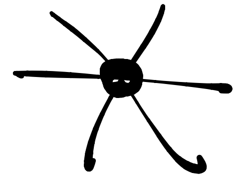
$$\delta = \delta_{N-1} + g_P^N \Delta\delta_N + \mathcal{O}(g_P^{N+1})$$

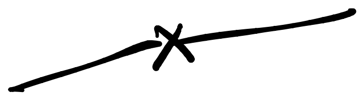
$\uparrow$ demanding that pole is at m_P^2 w/ residue 1 ...

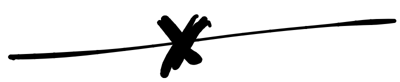
How do we know Λ doesn't reappear in, say, $\mathcal{M}_{\phi^3 \leftarrow \phi^3}$?



$$\delta\mathcal{L} = -\frac{\delta_6 \phi^6}{6!} \longrightarrow$$  $= \underline{\underline{\delta_6}}$

$i\mathcal{M}_{3\leftarrow 3} =$  $+$ 

$\longrightarrow$  $= \underline{\underline{\delta m^2}}$

 $= \delta^2(x_\perp)$

$$Z = \int D\phi \; e^{-S}$$

$$S = \int \left((\partial\phi)^2 + V \right)$$