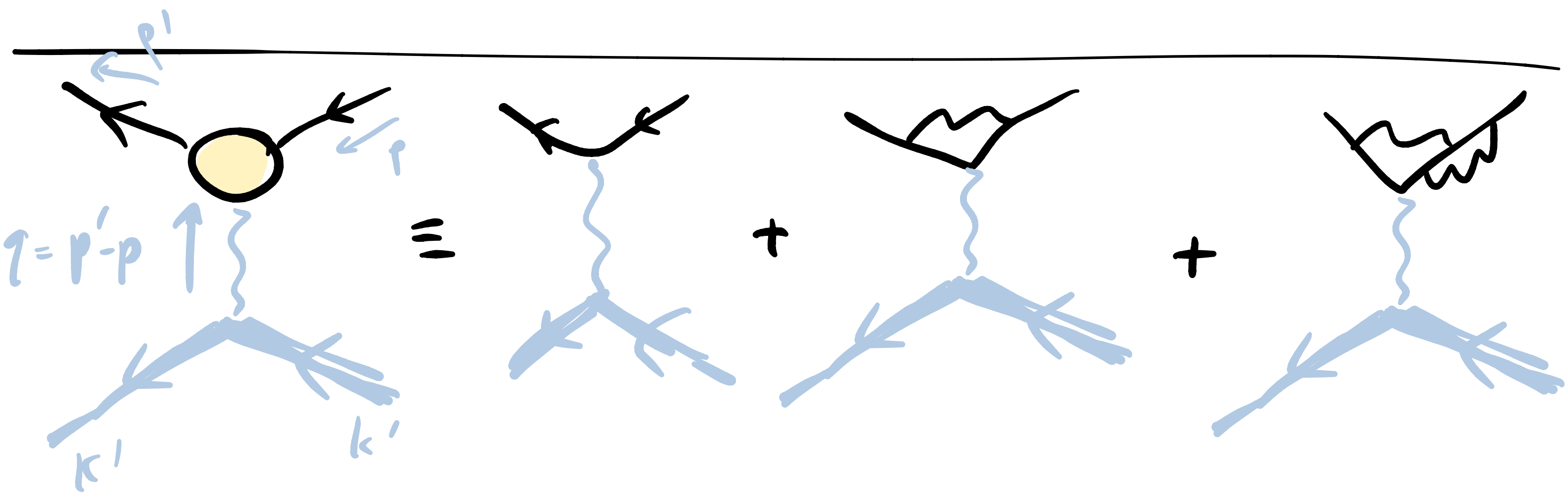
16 Vertex Correction

- cancel the UV divergence from δZ
- compute $g-2$ (at one loop)
- $$\left(\frac{d\sigma}{d\Omega}\right)_{e\mu\leftarrow e\mu} \simeq \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \underbrace{\left(1+\alpha(-\infty)+\mathcal{O}(\alpha^2)\right)}_{e^{-\alpha\infty}=0.}$$



$$\equiv i\mathcal{M} = ie^2\, \bar{u}(p')\Gamma^\mu(p,p')u(p)\,\frac{1}{q^2}\,\bar{u}(k')\gamma_\mu u(k)$$

'vertex function'

assume $p^2 = p'^2 = m_e^2$ but $q^2 \neq 0$. $q^2 = 2m_e^2 - 2p'\cdot p$.

$\Gamma^\mu = \gamma^\mu + \mathcal{O}(e^2)$

$\Gamma^\mu(p,p')$ is a vector made from:

$p^\mu \quad p'^\mu \quad \gamma^\mu \qquad m_e, e.$

no: γ^5, $\epsilon^{\mu\nu\rho\sigma}$ by

parity sym. of QED.

<u>general form:</u> $\circledast$ $\Gamma^\mu(p,p') = A\gamma^\mu + B(p+p')^\mu + C(p-p')^\mu$

A, B, C f'ns of $p^2 = (p')^2 = m_e^2$, $p \cdot p'$ ~~$\not{p}, \not{p}'$.~~

$\not{p}\gamma^\mu u(p)$

$= (m\gamma^\mu - p^\mu)u(p)$

$\Rightarrow$ <u>A, B, C f'ns of q^2.</u>

Ward

$$0 = q_\mu \bar{u}(p')\Gamma^\mu u(p) \overset{*}{=} \bar{u}(p')\left[A\cancel{\not{q}} + B\cancel{(p+p')(p-p')} + Cq^2\right]u(p)$$

$\bar{u}(p')(\not{p}'-\not{p})u(p)$

$= m - m = 0.$

$0 \qquad m_e^2 - m_e^2 = 0$

$\Rightarrow C = 0.$

Gordon identity: $\bar{u}(p')\gamma^\mu u(p) =$

$$\bar{u}(p')\left(\frac{p^\mu + p'^\mu}{2m} + \frac{i\sigma^{\mu\nu} q_\nu}{2m}\right)u(p)$$

$$\left(\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]\right)$$

$\Longrightarrow$

$$\Gamma^\mu(p,p') = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2)$$

form factors

$eF_1(q^2=0) =$ electric charge of electron.

if F_1 is renormalized we'd better

include $\mathcal{L}_{ct} = \underline{\delta_e}\, \bar{\psi}\gamma^\mu A_\mu \psi$

Renormalization cond: $\underline{F_1(0) \stackrel{!}{=} 1}$.

magn. dipole moment of e : $\vec{\mu}$ in

$$\tilde{V}_{eff}(q) = -\vec{\mu}\cdot\tilde{\vec{B}}(q)$$

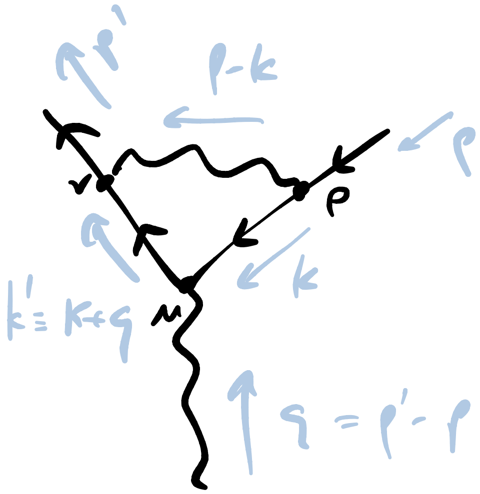
$$\bar{u}(p')\Gamma^i u(p) A_i(q) = -\vec{\mu}\cdot\vec{B}(q) + \dots$$

$$\Rightarrow \vec{\mu} = g\frac{e}{2m}\vec{S}$$

$$\leadsto \vec{S} = \xi^\dagger \frac{\vec{\sigma}}{2}\xi .$$

$$\Rightarrow g = 2\left(F_1(0) + F_2(0)\right)$$

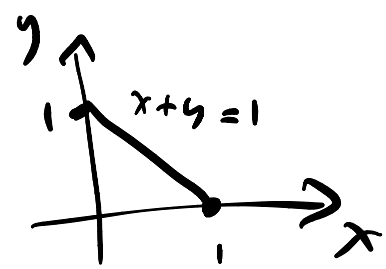
$$= 2 + \underbrace{2F_2(0)}_{\text{anomalous magn. moment.}} = \underline{\underline{2 + O(\alpha)}}$$



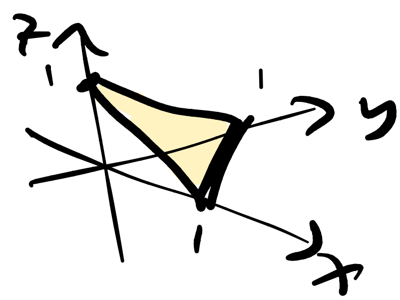
$$= (-ie)^3 \int d^4k \cdot \frac{-\eta_{\nu\rho}}{(p-k)^2 - m_\gamma^2} \cdot \bar{u}(p')\left[\gamma^\nu \frac{\not{k}' + m_e}{k'^2 - m_e^2}\gamma^\mu \frac{\not{k} + m_e}{k^2 - m_e^2}\gamma^\rho\right] u(p)$$

step 1 :

$$\frac{1}{AB} = \int_0^1 dx \int_0^1 dy \, \delta(x+y-1) \frac{1}{(xA+yB)^2}$$



$$\frac{1}{ABC} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \, \delta(x+y+z-1) \frac{2!}{(xA+yB+zC)^3}$$



Pf : $\frac{1}{A} = \int_0^\infty ds \, e^{-sA}$

$$\frac{1}{A_1 \cdots A_n} = \int_0^\infty ds_1 \cdots \int_0^\infty ds_n \, e^{-\sum_{i=1}^n s_i A_i}$$

$$\begin{cases} \tau \equiv \sum_{i=1}^n s_i \\ x_i \equiv s_i/\tau \end{cases}$$

$$= \int_0^\infty d\tau \, \tau^{n-1} \prod_{i=1}^n \int_0^1 dx_i \, \delta\left(\sum_{i=1}^n x_i - 1\right) \exp\left(-\tau \sum_i x_i A_i\right)$$

$$\int_0^\infty d\tau\, \tau^{n-1} e^{-\tau X} = \frac{(n-1)!}{X} \Leftarrow \left(\begin{array}{l} \text{differentiate} \\ \frac{1}{A} = \int ds\, e^{-sA} \\ \text{wrt } A. \end{array} \right)$$

$$\Rightarrow \frac{1}{A_1 \cdots A_n} = \prod_{i=1}^{n} \int_0^1 dx_i\, \delta\left(\sum_{j=1}^{n} x_j - 1\right) \frac{(n-1)!}{\left(\sum_i x_i A_i\right)^n}$$

■

$$\text{set} \begin{cases} A = (k')^2 - m^2 + i\epsilon \\ B = k^2 - m^2 + i\epsilon \\ C = (p-k)^2 - m_\gamma^2 + i\epsilon \end{cases}$$

$$\rightarrow \int \frac{d^4k\, N^\mu}{\left(k^2 + k\cdot\left(2(xq - zp)\right) + \circledcirc\right)^3}$$

step 2: complete the square $\ell = k - zp + xq$

$$= \int \frac{d^4\ell\, N^\mu}{(\ell^2 - \Delta)^3}$$

$$\Delta = -xyq^2 + (1-z)^2 m^2 + z m_\gamma^2 .$$

$$N: \quad 1 \qquad\qquad\qquad 1$$

$$\ell^\mu \xrightarrow[\int_{S^3} d\Omega_\ell]{\text{average over}} 0$$

$$\ell^\mu \ell^\nu \qquad\qquad \frac{1}{4}\ell^2 \eta^{\mu\nu}$$

Step 3: wick rotate.

$$\int \frac{d^D \ell}{(\ell^2-\Delta)^m} = (-1)^m \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(m-D/2)}{\Gamma(m)} \left(\frac{1}{\Delta}\right)^{m-D/2}$$

$$\int \frac{d^D \ell \, \ell^2}{(\ell^2-\Delta)^m} = (-1)^{m-1} \frac{D}{2} \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(m-\frac{D}{2}-1)}{\Gamma(m)} \left(\frac{1}{\Delta}\right)^{m-\frac{D}{2}-1}.$$

$$\sim \int^\Lambda \frac{d\ell \, \ell^{D+1}}{\ell^{2m}} \qquad = \infty \text{ for } D=4 \;\; m=3.$$

$$\sim \Lambda^{D+2-2m} \xrightarrow[m=3]{D=4} \Lambda^0 \longrightarrow \text{Log}.$$

Step 0: use a PV regulator

$$\text{ans}(m_\gamma) \rightsquigarrow \text{ans}(m_\gamma) - \text{ans}(\Lambda)$$

$$N^\mu = \bar{u}(p')\gamma^\nu(\not{k}+\not{q}+m)\gamma^\mu(\not{k}+m)\gamma_\nu u(p)$$

$$= -2\Big(A\,\underbrace{\bar{u}(p')\gamma^\mu u(p)}_{\text{renormalize } e} + B\,\underbrace{\bar{u}(p')\sigma^{\mu\nu}q_\nu u(p)}_{\text{magn. moment}}$$

$$+ C\,\underbrace{\bar{u}(p')q^\mu u(p)}_{\text{zero!}}\Big)$$

$$\begin{cases} A = -\frac{1}{2}\ell^2 + (1-x)(1-y)q^2 + (1-4z+z^2)m^2 \\ B = i m z(1-z) \\ C = m(z-2)(y-x) \end{cases}$$

odd under $x \leftrightarrow y$

$$\int dx\,dy\,dz\,\delta(1-x-y-z)\,C = 0. \checkmark$$

cutoff dependence is ONLY in A:

$$\int d^4 l \left(\frac{l^2}{(l^2 - \Delta_{m_\gamma})^3} - \frac{l^2}{(l^2 - \Delta_\Lambda)^3} \right)$$

$$= \frac{i}{(4\pi)^2} \log \frac{\Delta_\Lambda}{\Delta_{m_\gamma}}$$

Anomalous Magnetic Moment:

$$F_2(q^2) = \frac{2m}{e} \left(\text{the term } B \right)$$

$$= \frac{2m}{e} 4e^3 (im) \int dx\,dy\,dz\, \delta(1-x-y-z)\, z(1-z) \int \frac{d^4 l}{(l^2 - \Delta)^3}$$

$$\int \frac{d^4 l}{(l^2 - \Delta)^3} = \frac{-i}{32\pi^2 \Delta}$$

$m_\gamma = 0$

$$\downarrow$$

$$= \frac{\alpha}{\pi} m^2 \int dx\,dy\,dz \frac{\delta(1-x-y-z)\, z(1-z)}{(1-z)^2 m^2 - xyq^2}$$

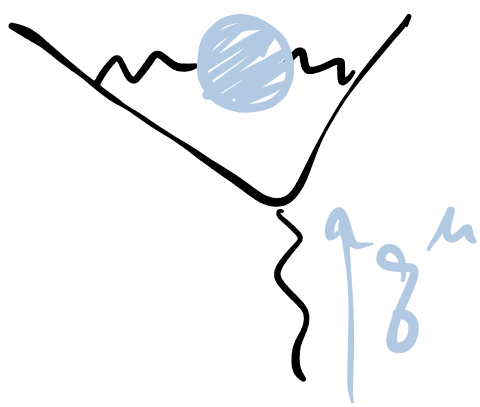
$$\frac{g-2}{2} = F_2(q^2=0) = \frac{\alpha}{\pi} m^2 \int_0^1 dz \int_0^{\cancel{1-z}} dy \cdot \frac{z}{\cancel{(1-z)}\, m^2}$$

$$= \frac{\alpha}{2\pi}.$$

$$\Rightarrow \quad g = 2 + \frac{\alpha}{\pi} + O(\alpha^2)$$

$$= 2.00232 + O(\alpha^2)$$

$$g_{(\mu)}^{expt} = 2.00233184121(82)$$



IR divergences mean wrong questions

$A\gamma^\mu$ bit:

$$\int \frac{d^4 l}{(l^2-\Delta)^2} = \frac{c}{\Delta} \qquad \left(c = \frac{-1}{32\pi^2}\right)$$

$$\int dx\,dy\,dz\,\delta(1-x-y-z) \int \frac{d^4 l\, A}{(l^2-\Delta)^2}$$

$q^2=0,\ m_\gamma=0.$

$$= c\int dx\,dy\,dz\,\delta(1-x-y-z)\, m^2 \frac{(1-4z+z^2)}{\Delta}$$

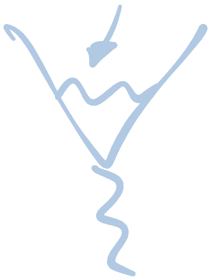
near $z \to 1$

$$= m^2 \int^1 dz \underbrace{\int_0^{1-z} dy}_{1-z} \frac{-2+2(1-z)+(1-z)^2}{(1-z)^2 m^2 + m_\gamma^2 z}$$

$$= -2\int^1 dz \frac{1}{(1-z)} + \text{finite} \sim \int \frac{dz\,(1-z)}{(1-z)^2 m^2 + m_\gamma^2 z} < \infty$$

IR singular bit of Γ (blows up when $m_\gamma \to 0$):

$$\Gamma^\mu = \gamma^\mu \left(1 - \frac{\alpha}{2\pi} f_{IR}(q^2) \ln\left(\frac{-q^2}{m_\gamma^2}\right)\right)$$



+ stuff that's finite when $m_\gamma \to 0$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mu e \leftarrow \mu e} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \left(1 - \underbrace{\frac{\alpha}{\pi} f_{IR}(q^2) \ln\left[\frac{-q^2}{m_\gamma^2}\right]}_{\to -\infty \text{ when } m_\gamma \to 0}\right) + \mathcal{O}(\alpha^2)$$

for t-channel exchange

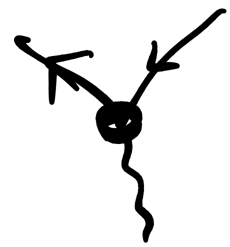
$q^2 < 0$

$\ln\left(\frac{-q^2}{m_\gamma^2}\right) > 0.$

$1 - \alpha\infty \sim e^{-\alpha\infty} \to 0.$

$$F_1(q^2) = 1 + \underline{f(q^2)} + \delta_e + \mathcal{O}(\alpha^2)$$

$\uparrow$ tree $\qquad$ bit from A $\qquad$ $\uparrow$ counterterm for



$$f(q^2) = \frac{e^2}{8\pi^2}\int dx\,dy\,dz\,\delta(1-x-y-z)\left(\ln\frac{z\Lambda^2}{\Delta} + \frac{q^2(1-x)(1-y) + m_e^2(1-4z+z^2)}{\Delta}\right)$$

Consider: $-q^2 \gg m_e^2$

$$1 \overset{!}{=} F_1(0) \Rightarrow \delta_e = -f(0)$$

$$\xrightarrow{m_e/q \to 0} -\frac{e^2}{8\pi^2}\frac{1}{2}\ln\frac{\Lambda^2}{m_\gamma^2}.$$

$$f(q^2)\Big|_{m_e=0} = \frac{e^2}{8\pi^2}\int dx\,dy\,dz\,\delta(\;)\Bigg(\underbrace{\ln\frac{(1-x-y)\Lambda^2}{\Delta}}_{\text{IR finite}} + \frac{q^2(1-x)(1-y)}{-xyq^2+(1-x-y)m_\gamma^2}\Bigg)$$

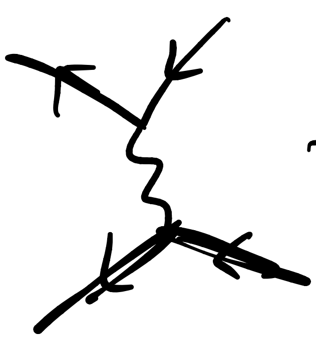
$$\boxed{q^2 \to q^2 + i\epsilon}$$

$$\Rightarrow F_1(q^2)\Big|_{m_e=0} = 1 - \frac{e^2}{16\pi^2}\left(\left(\ln\frac{-q^2}{m_\gamma^2}\right)^2 + 3\ln\frac{-q^2}{m_\gamma^2}\right) + \text{finite}.$$

'Sudakov double logarithm'.

cannot be removed by taking differences.

EDI to the rescue: we can't distinguish



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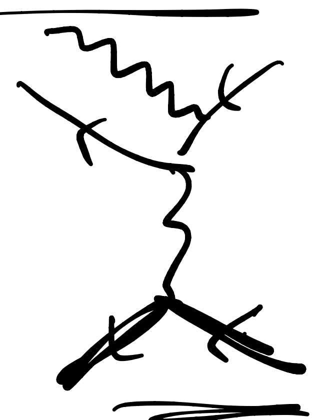
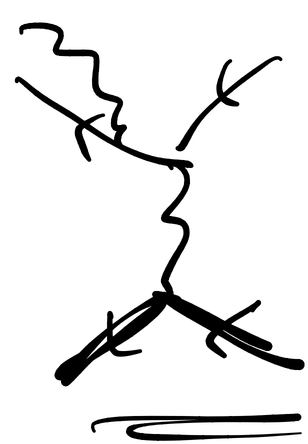
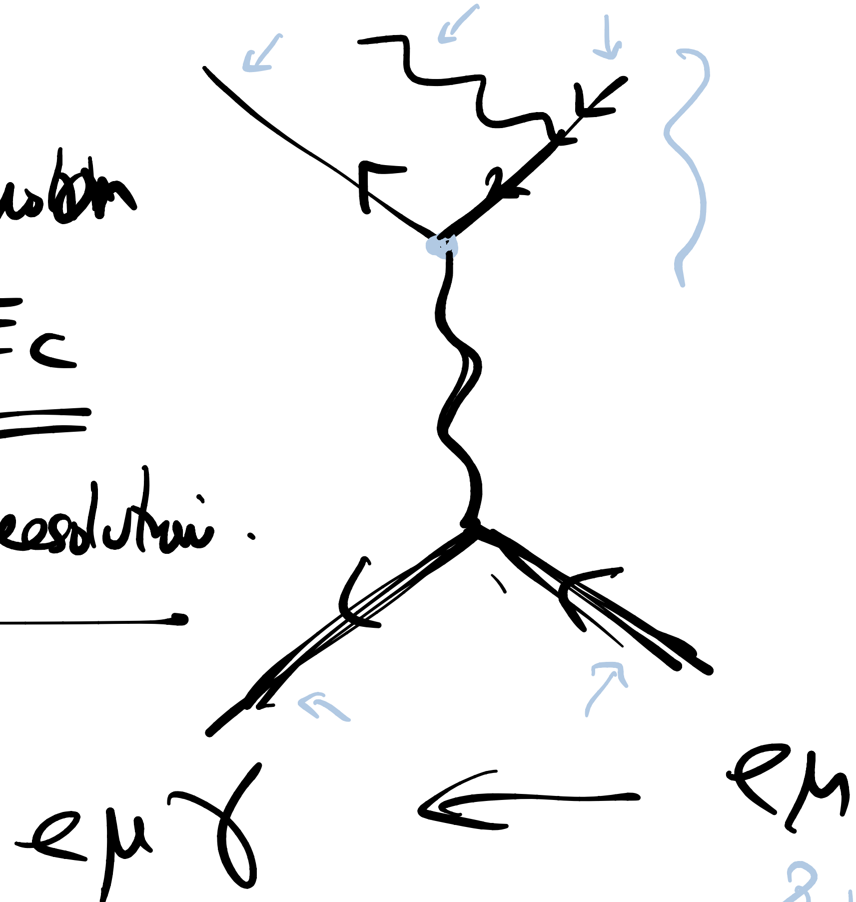


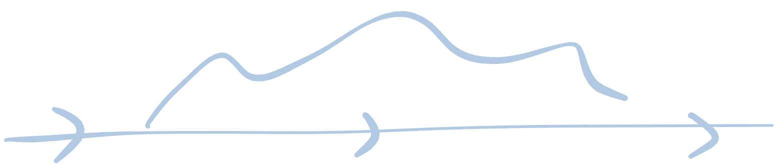
OR

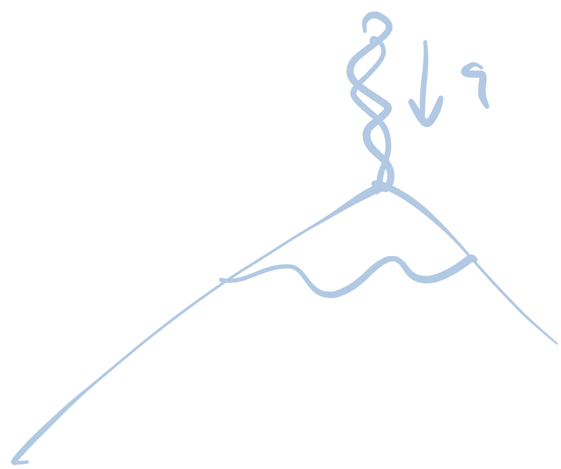


If the energy of the extra photon is $E < E_c$

detector resolution.







$G(p) =$ 

$$= \frac{i}{p^2 - \underbrace{m_0^2 - \Sigma(p)}} = \frac{i}{p^2 - m_{eff}^2}$$

if $\Sigma(p^2) = \Sigma(0)$

Suppose

$$G^{-1}(p) = p^2 - m_0^2 - \Sigma(p^2) = 0 \quad \text{when } \underline{\underline{p^2 = m^2}}$$

$$\Rightarrow m^2 = m_0^2 + \Sigma(m^2).$$

$$G^{-1}(p) = p^2 - \underbrace{m_0^2 - \big(\Sigma(m^2)} + (p^2 - m^2)\Sigma'(m^2) + \dots\big)$$

$$= p^2 - m^2 - (p^2 - m^2)\Sigma'(m^2) + O\big((p^2 - m^2)^2\big)$$

$$= (p^2 - m^2)\big(1 - \Sigma'(m^2)\big) + \dots$$

$$G(p^2) \overset{p^2\to m^2}{\sim} \frac{i}{p^2-m^2}\underline{\underline{\left(\frac{1}{1-\Sigma'(m_0^2)}\right)}}$$

$$\mathcal{L} = \underline{\underline{f}}\frac{(\partial\phi)^2}{2} + \frac{m^2}{2}\phi^2$$

$$G(p^2) = \frac{i}{fp^2-m^2}$$

① $G^{-1}(p^2)\Big|_{m^2} \overset{!}{=} 0$ requires $\delta_{m^2}\phi^2$

② $G(p^2) \overset{p^2\to m^2}{\sim} \frac{i}{p^2-m^2}$ requires $\delta_Z(\partial\phi)^2$

Relevant $\longleftrightarrow$ changes low T behavior.

(Irrel) $\longleftrightarrow$ (doesn't)

$$\int dx\, e^{-\frac{1}{T}(x^4 + \epsilon x^{2n})}$$

$$X \equiv \frac{x}{T^{1/4}} \qquad x = T^{1/4} X$$

$$= T^{+1/4} \int dX\, e^{-X^4 - \underbrace{\frac{\epsilon}{T} T^{\frac{2n}{4}} X^{2n}}}$$

changes the low T behav

if $\frac{2n}{4} - 1 > 0$

$$\delta V = X^{2n} \cdot Y^{2m}$$