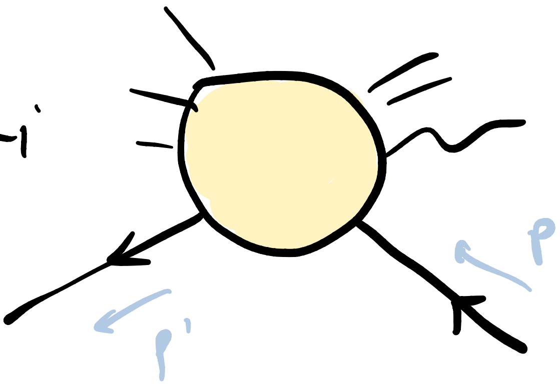
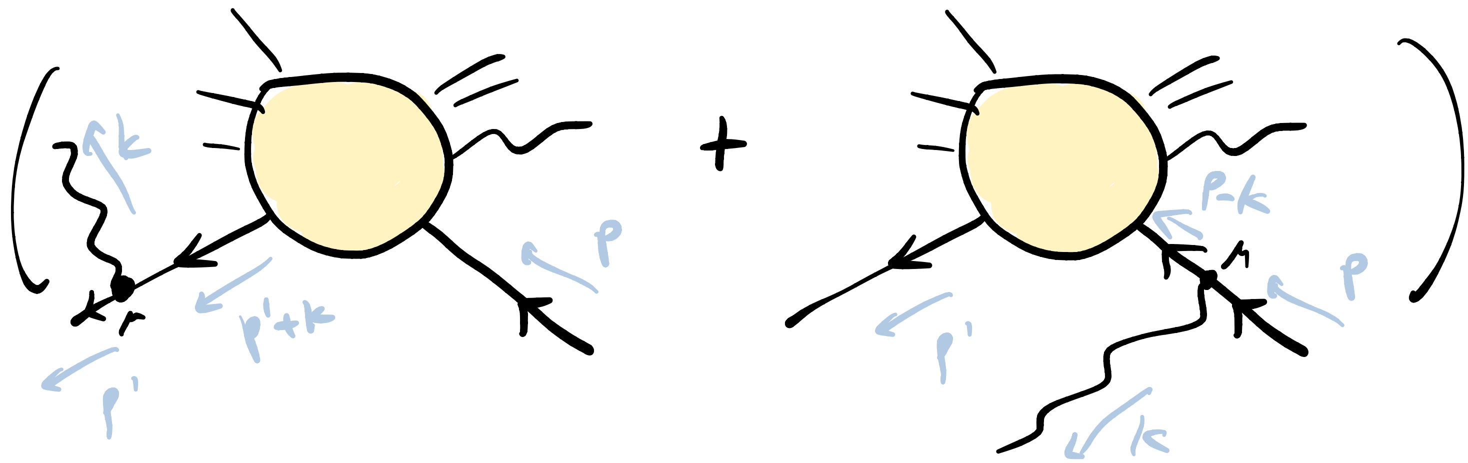
1.6 Soft photons

Because $m_\gamma = 0$, in any process w/ external charges, we can't distinguish

$-i$ 

$= \bar{u}(p')\mathcal{M}_0(p',p)u(p)$

from more inclusive processes:

$-i$ 

$$= \bar{u}(p')\gamma^\mu \frac{e}{\not{p}'+\not{k}-m_e}\mathcal{M}_0(p',p)u(p)\epsilon^*_\mu(k)$$

$$+\bar{u}(p')\mathcal{M}_0(p',p)\frac{e}{\not{p}-\not{k}-m_e}\gamma^\mu u(p)\epsilon^*_\mu(k)$$

where $\underline{\underline{k^0 < E_c}}$ — in the rest frame of the detector.

$\underline{\text{soft}}$: $(\not{p} - \not{k} + m_e)\gamma^\mu u(p) \simeq (\not{p} + m_e)\gamma^\mu u(p)$

$$\overset{\text{Clifford}}{=} \left(2p^\mu + \gamma^\mu \underbrace{(-\not{p} + m_e)}_{\to 0}\right) u(p)$$

$$= 2p^\mu u(p).$$

$$(p-k)^2 - m_e^2 = \cancel{p^2 - m_e^2}^{\to 0} - 2p\cdot k + \cancel{k^2}^{\to 0} \quad \underline{\text{on-shell photon}}$$

$$= -2p\cdot k.$$

$$\mathcal{M}(e\underline{\mu} + \text{one soft photon} \leftarrow e\underline{\mu}) = e\bar{u}(p')\mathcal{M}_0(p',p)u(p) \times \left(\frac{p'^\mu}{p'\cdot k + i\epsilon} - \frac{p^\mu}{p\cdot k - i\epsilon}\right)\epsilon^*_\mu$$

This is $\underline{\text{bremsstrahlung}}$:

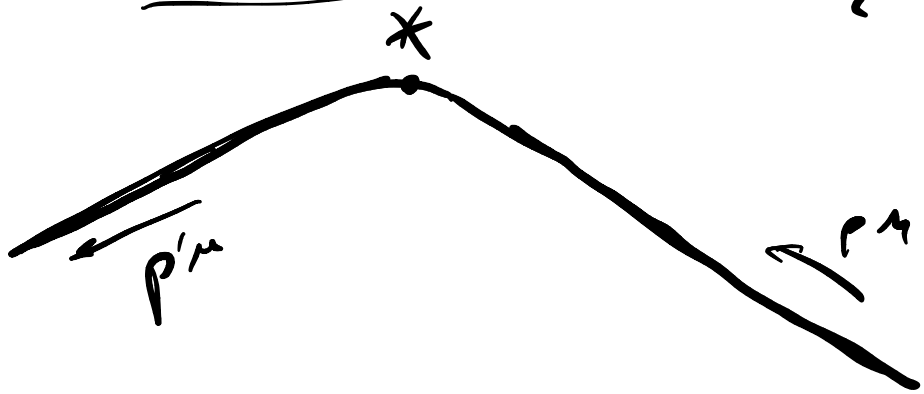
$$\frac{j^\mu(k)}{ie} \equiv \frac{p'^\mu}{p'\cdot k + i\epsilon} - \frac{p^\mu}{p\cdot k - i\epsilon}$$

current for a charged particle follows $x^\mu = y(\tau)^\mu$ is

$$j^\mu(x) = e \int d\tau \, \frac{dy^\mu}{d\tau} \delta^{(4)}(x - y(\tau))$$

If we set:

$$y^\mu(\tau) = \begin{cases} \frac{p^\mu}{m}\tau & \tau < 0 \\ \frac{p'^\mu}{m}\tau & \tau > 0 \end{cases}$$



sudden acceleration.

$$\tilde{j}^\mu(k) \equiv \int d^4x \, e^{ikx} j^\mu(x)$$

Maxwell's eqn: $$\tilde{A}^\mu(k) = -\frac{1}{k^2}\tilde{j}^\mu(k)$$

$$\rightsquigarrow U = \frac{1}{2}\int d^3x \left(E^2 + B^2\right)$$

$$= \int d^3k \, \omega_k n_k$$

# of photons w/ wave # k.

# of photons produced / decade of wave #

is $f_{IR}(q^2) = \frac{\alpha}{\pi} \ln\left(\frac{-q^2}{m^2}\right)$.

$\left(q = p' - p \;,\; -q^2 \geq 0. \right)$

$$\left(\frac{d\sigma}{d\Omega}\right)^{E_\gamma < E_c}_{\mu e \gamma \leftarrow \mu e} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott} \cdot e^2 \underbrace{\int_0^{E_c} \frac{d^3k}{2E_k^\gamma}}_{\text{final state phase space of } \gamma} \left| \frac{p\cdot\epsilon^*}{p\cdot k} - \frac{p'\cdot\epsilon^*}{p'\cdot k} \right|^2$$

SAME IR Regulator:

$E_k = \sqrt{\vec{k}^2 + m_\gamma^2}$

If $m_\gamma = 0$

$E_k = |k|$

$$\simeq \int_0 \frac{d^3k}{k^3} = \infty .$$

$\overset{\text{FAKE}}{m_\gamma} << \overset{\text{REAL}}{E_c}$

another IR divergence

$$\int_0^{E_c} \frac{dk}{E_k} = \int_0^{m_\gamma} + \int_{m_\gamma}^{E_c} \frac{dk}{\sqrt{k^2 + m_\gamma^2}} \cong \underbrace{\int_0^{m_\gamma} \frac{dk}{m_\gamma}}_{1} + \underbrace{\int_{m_\gamma}^{E_c} \frac{dk}{k}}_{\log\left(\frac{E_c}{m_\gamma}\right)!}$$

$$\left(\frac{d\sigma}{d\Omega}\right)^{\text{observed}} = \left(\frac{d\sigma}{d\Omega}\right)_{e\mu\leftarrow e\mu} + \left(\frac{d\sigma}{d\Omega}\right)^{E_\gamma < E_c}_{e\mu\gamma_{soft}\leftarrow e\mu} + \mathcal{O}(\alpha^3)$$

option ① $\int d\Pi \left| A_{e\mu\leftarrow e\mu} + A_{e\mu\gamma\leftarrow e\mu} \right|^2$

option ② ✓ $d\sigma_{e\mu\leftarrow e\mu} + d\sigma_{e\mu\gamma\leftarrow e\mu}$

processes in different final states add incoherently.

$$= \left(\frac{d\sigma}{d\Omega}\right)_{Mott}\left[\overset{\text{tree}}{1} \underbrace{- \frac{\alpha}{\pi} f_{IR}(q^2)\ln\left(\frac{-q^2}{m_\gamma^2}\right)}_{\text{vertex correction}} + \underbrace{\frac{\alpha}{\pi} f_{IR}(q^2)\ln\left(\frac{E_c^2}{m_\gamma^2}\right)}_{\text{one soft photon}}\right] + \mathcal{O}(\alpha^3)$$

$$= \left(\frac{d\sigma}{d\Omega}\right)_{Mott}\left[1 - \frac{\alpha}{\pi} f_{IR}(q^2)\ln\left(\frac{-q^2}{E_c^2}\right)\right] + \mathcal{O}(\alpha^3).$$

n-loop vertex correction $\sim \ln^n(q^2/m_\gamma^2)$

$\uparrow$ leading IR singularity

amplitude to emit n soft γ's $\sim -\ln^n(E_c^2/m_\gamma^2)$

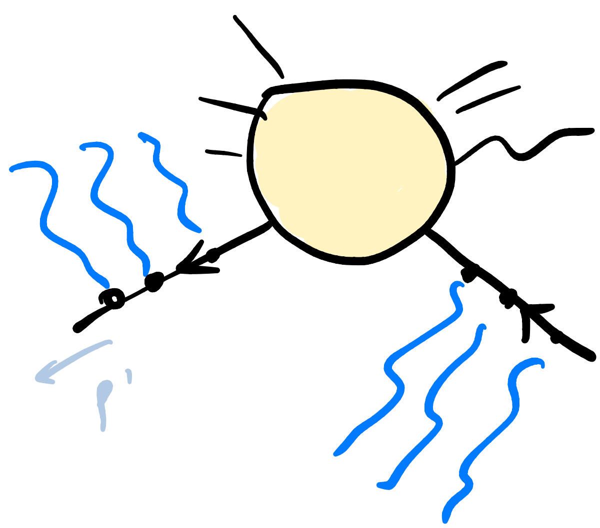
$$\longrightarrow e^{-\frac{\alpha}{\pi} f \ln(-q^2/m_\gamma^2) + \frac{\alpha}{\pi} f \ln(E_c^2/m_\gamma^2)}$$

$$= e^{-\frac{\alpha}{\pi} f \ln(-q^2/E_c^2)}$$

(Bloch-Nordsieck Thm.)

sketch of pf:

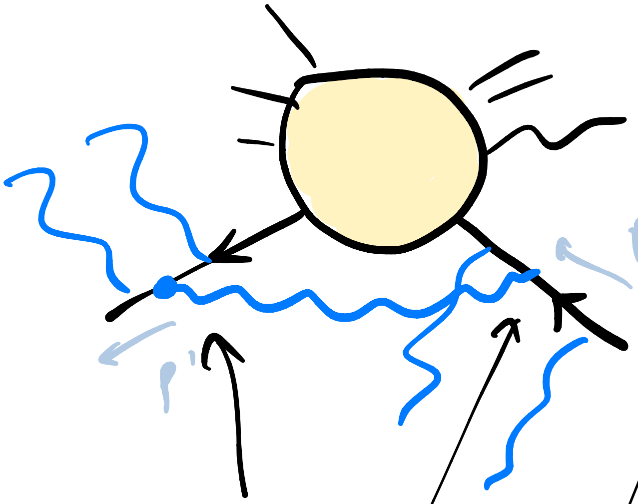
n soft photons $\{k_\alpha\}$

$\sum_{n_f=0}^{n}$  $\equiv A_n$

final state rad

initial state rad

$$= \bar{u}(p')\, i\mathcal{M}_0\, u(p)\, e^n \prod_{a=1}^{n} \underbrace{\left(\frac{p'^{\mu_a}}{p'\cdot k_\alpha} - \frac{p^{\mu_a}}{p\cdot k_\alpha} \right)} \epsilon^*_{\mu_a}$$



$k_\alpha = -k_\beta = k$

just the IR singular

$$\simeq A_{n-2}\,\frac{e^2}{2}\int d^4k\;\frac{-i\eta_{\rho\sigma}}{k^2-m_\gamma^2}\;\underbrace{\left(\frac{p'}{p'\cdot k}-\frac{p}{p\cdot k}\right)^{\rho}\left(\frac{p'}{-p'\cdot k}-\frac{p}{-p\cdot k}\right)^{\sigma}}_{e^-\text{ propagators when } k \text{ is small.}}$$

$\alpha \leftrightarrow \beta$

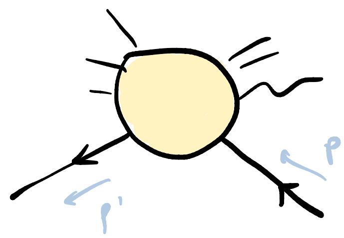
For $n=2$: $\longrightarrow$

$$\underbrace{\bar u\, i\mathcal{M}_0\, u}_{A_0}\; X \equiv$$

$$X = -\frac{\alpha}{2\pi}\, f_{IR}(q^2)\,\ln\!\left(-\frac{q^2}{m_\gamma^2}\right) + \text{finite when } m_\gamma \to 0.$$

$$\mathcal{M}_{\substack{\text{virtual}\\ \text{soft photons}}} = \sum_{m=0}^{\infty} \left(\right.$$  $$\left. \right) \leftarrow \text{LOOPS}$$

m of these

$$= \left(\right.$$  $$\left. \right) \cdot \sum_m \frac{1}{m!} X^m$$

$$= i\mathcal{M}_0 \, e^{X} \xrightarrow{m_\gamma \to 0} 0$$

since $X \xrightarrow{m_\gamma \to 0} -\infty$

(soft)
Real photons: $\Rightarrow d\sigma_{\text{exclusive}} \propto e^{2X} \to 0.$

ward $\to -\eta^{\mu\nu}$

$$d\sigma_{1\gamma} = \int d\Pi \sum_{\text{pols}} \epsilon^{\mu} \epsilon^{*\nu} \mathcal{M}_\mu \mathcal{M}^*_\nu$$

$$= \int d\Pi_0 \left| \bar{u}(p') \mathcal{M}_0 u(p) \right|^2 \int^{E_c} \frac{d^3k}{2E_k} (-\eta_{\mu\nu}) e^2 \left(\frac{p'}{p'\cdot k} - \frac{p}{p\cdot k} \right)^{\mu} \left(\frac{p'}{-p'\cdot k} - \frac{p}{-p\cdot k} \right)^{\nu}$$

$$\equiv d\sigma_0 \, Y$$

$$Y = \frac{\alpha}{\pi} f_{IR}(q^2) \log\left(\frac{E_c^2}{m_\gamma^2}\right)$$

$$d\sigma_{n\gamma} = \frac{1}{n!} d\sigma_0 Y^n$$

↑ n Bosons in final state

<u>observable</u>:

$$\sum_{n=0}^{\infty} d\sigma_{n\gamma} = d\sigma_0 \sum_n \frac{1}{n!} Y^n$$

$$= d\sigma_0 e^{Y}.$$

Actual cross section w both:

Leading IR singularity.

$$d\sigma \overset{\downarrow}{=} d\sigma_0 e^{2X} e^{Y}$$

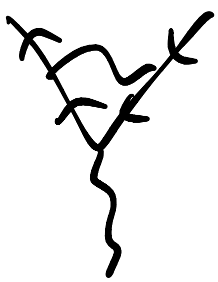
$$= d\sigma_0 \exp\left[-\frac{\alpha}{\pi} f_{IR}(q^2) \ln\left(\frac{-q^2}{m_\gamma^2}\right) + \frac{\alpha}{\pi} f_{IR}(q^2) \ln\left(\frac{E_c^2}{m_\gamma^2}\right)\right]$$

$$= d\sigma_0 \exp\left[-\frac{\alpha}{\pi} f_{IR}(q^2) \ln\frac{-q^2}{E_c^2}\right].$$

Some apparent magic:

$$\rightsquigarrow \quad Z_e = 1 + \left.\frac{\partial \Sigma}{\partial \not p}\right|_{\not p = m_0} + O(e^4)$$

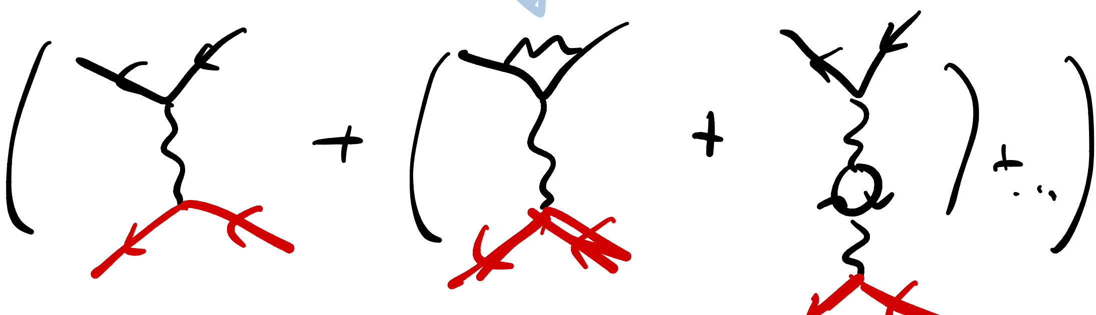
$$= 1 - \underline{\underline{\frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2}}} + \text{finite} + O(\alpha^2)$$



$$\rightsquigarrow \quad \underline{\Gamma^\mu = e\gamma^\mu \underline{F_1(q^2)}} + \dots$$

$$F_1(q^2) = 1 + \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} + \text{finite} + O(\alpha^2)$$

$$S_{e\mu \leftarrow e\mu} = \left(\sqrt{Z_e}\right)^2 \Big(\quad + \quad + \quad \Big) + \dots$$



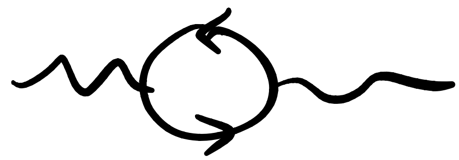
$$= \left(1 - \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} + \dots\right) e^2 \bar{u}(p') \left[\gamma^\mu \left(1 + \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} + \dots\right) + \alpha \frac{i\sigma^{\mu\nu} q_\nu}{2m}\right] u(p) \dots$$

$$= \left(1 - \cancel{\frac{\alpha}{4\pi}\ln\frac{\Lambda^2}{m^2}} + \cancel{\frac{\alpha}{4\pi}\ln\frac{\Lambda^2}{m^2}} + \mathcal{O}(\alpha^2)\right)$$

$$\times\ e^2\ \bar{u}(p')\ \gamma^\mu\ u(p)\ +\ \text{finite} + \mathcal{O}(\alpha^3)$$

why did this happen?

1.7 Vacuum Polarization



$$\mathcal{L}_{QED} = \bar{\psi}(\not\partial + ie\,\not{\tilde{A}} - m)\psi - \frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$$

$$\tilde{F} \equiv d\tilde{A}$$

let $\begin{cases} e\tilde{A}_\mu \equiv A_\mu. \\ e\tilde{F}_{\mu\nu} \equiv F_{\mu\nu} \end{cases}$

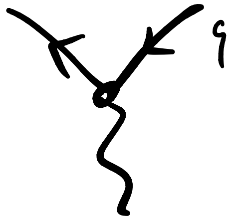
$$\rightarrow \mathcal{L}_{QED} = \bar{\psi}(\not\partial + i\not{A} - m)\psi - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}$$

$$\begin{cases} A_\mu \rightarrow A_\mu + \partial_\mu\lambda \\ \psi_q \rightarrow e^{iq\lambda}\psi_q \end{cases} \qquad \left(\tilde{A} \rightarrow \tilde{A} - \frac{\partial\lambda}{e}\right)$$

$q \in \mathbb{Z}$.

$$\begin{cases} q_e = -1 \\ q_p = +1 \end{cases}$$

$$\langle A_\mu A_\nu \rangle \sim \frac{-i\eta_{\mu\nu} e^2}{q^2}$$ (has two ends on vertices)



$\sim -i\gamma^\mu q$

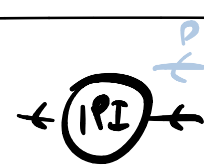
the magic was just gauge invariance

gauge inv. relates the coeff of

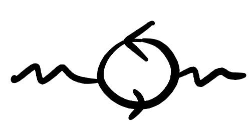
$\bar\Psi \not{A} \Psi$ to that of $\bar\Psi \not\partial \Psi$

$$\left(D_\mu \Psi \equiv (\partial + q i A)_\mu \Psi \longrightarrow e^{iq\lambda} D_\mu \Psi \right)$$

only $\bar\Psi \not{D} \Psi$ is gauge inv't

$-i\Sigma(p) =$ 

$+i\Pi_{\mu\nu}(q^2) =$ 

$=$  $+ \mathcal{O}(e^4)$

$S(\not p) = \frac{i}{\not p - m}$ but $\sim = \frac{-i\eta_{\mu\nu}}{p^2 - m_\gamma^2}$

<u>Lorentz</u> $\Rightarrow$ $\Pi^{\mu\nu}(q^2) = A(q^2)\eta^{\mu\nu} + B(q^2)q^\mu q^\nu$

Ward $\Rightarrow$ $0 = q_\mu \Pi^{\mu\nu}(q)$

$$\Rightarrow A q^\nu + B q^2 q^\nu = 0$$

$$\Rightarrow B = -A/q^2 .$$

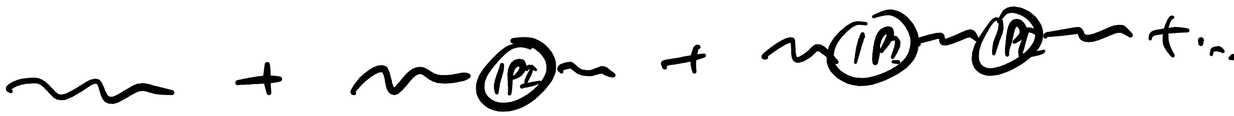
let $A \equiv \Pi q^2$

$$\Pi^{\mu\nu}(q^2) = \Pi(q^2) \cdot q^2 \underbrace{\left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)}_{\Delta_T^{\mu\nu}}$$

Δ_T is a <u>projector</u> onto vectors $\perp$ to q^μ

$$(\Delta_T)^\mu{}_\nu (\Delta_T)^\nu{}_\rho = (\Delta_T)^\mu{}_\rho .$$

 $= \dfrac{-i\Delta_T}{q^2}$ works by Ward id

$\tilde{G}(p) =$ 

$$= \frac{-i\Delta_T}{q^2}\left(1 + i\Pi \cancel{q^2}\Delta_T\left(\frac{-i\Delta_T}{\cancel{q^2}}\right) + i\Pi q^2\Delta_T\left(\frac{-i\Delta_T}{q^2}\right)\left(i\Pi q^2\Delta_T\right)\left(\frac{-i\Delta_T}{q^2}\right) + \dots\right)$$

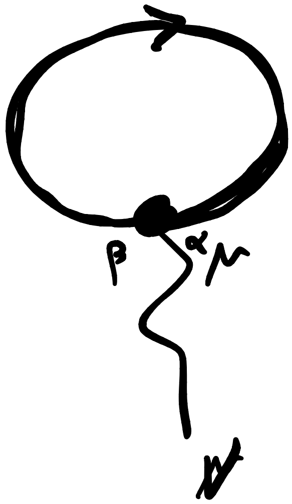
$\Delta_T^2 = \Delta_T$

$$= \frac{-i\Delta_T}{q^2}\left(1 + \Pi\Delta_T + \Pi^2\Delta_T + \dots\right)$$

$$= \frac{-i\Delta_T}{q^2}\,\frac{1}{1-\Pi(q^2)} = \frac{-i\Delta_T}{q^2 - q^2\Pi(q^2)}.$$

<u>Q</u>: does the photon get a mass?

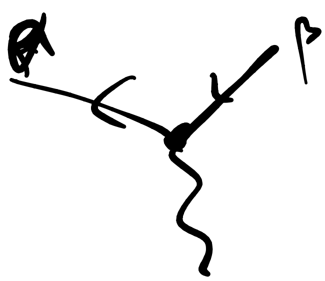
$$= \sum_{\alpha\beta} \left(\frac{1}{\not k - m_e}\right)_{\alpha\beta} \gamma^{\mu}_{\beta\gamma}$$



$$= \Lambda \, tr\left(\frac{\not k + m_e}{k^2 - m_e^2} (-ie)\gamma^{\mu}\right)$$

$$\frac{-i\eta_{\mu\nu}}{-m_{\mu}^2}$$

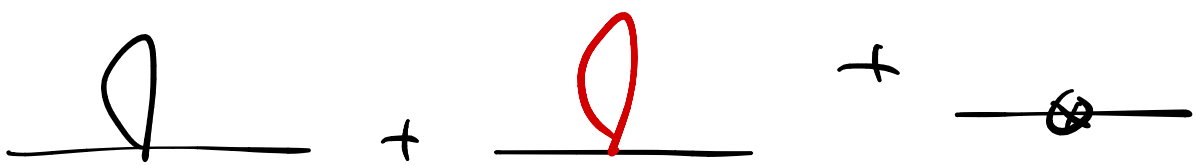
$= \; ....$

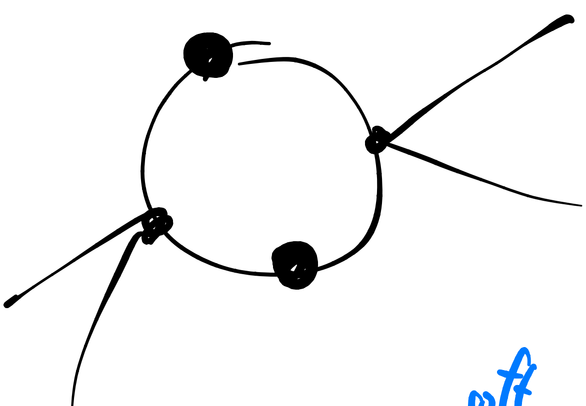


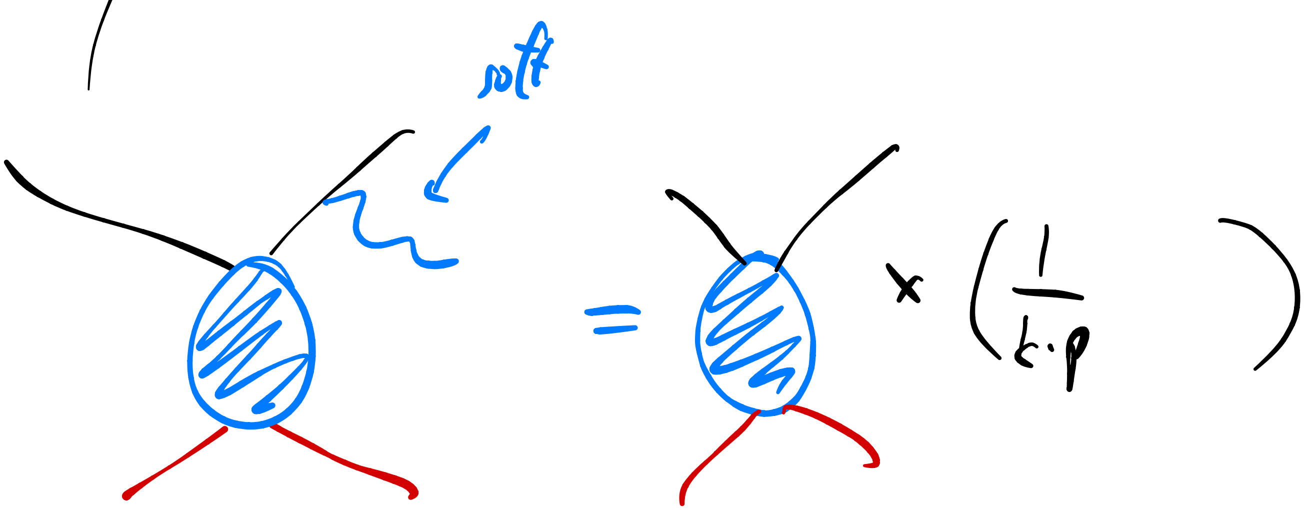
$$= -ie\gamma^{\mu}_{\alpha\beta}$$

$$G(p^2) \overset{!}{\sim} \frac{i\,1}{p^2 - m^2}\Bigg|_{m=0}$$

$$\mathcal{L}_{ct} \ni (\partial\phi_i)^2\,\delta Z + \delta m^2 \cdot \phi_i^2$$







$= \ \ldots \times \left(\frac{1}{k\cdot p}\right)$

$$\mathcal{M}_{n\,\text{initial}\,\gamma\text{'s}} = \mathcal{M}_0 \times X^n$$

$$d\sigma = \frac{\int d\Pi \int d\Pi_\gamma \; |\mathcal{M}_0|^2 X^n}{\int d\Pi_\gamma \; X^n}$$

$$= \int d\Pi \, |\mathcal{M}_0|^2$$