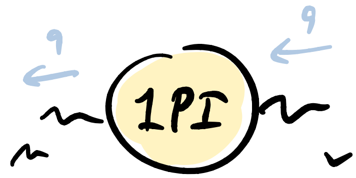
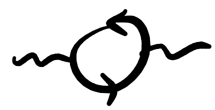
1.7 Vacuum Polarization, cont'd

$i\Pi_{\mu\nu}(q^2) \equiv$ 

$=$  $+ \mathcal{O}(e^4)$

Lorentz & gauge inv $\Longrightarrow$ $i\Pi_{\mu\nu}(q^2) = \Pi(q^2)\, q^2\, \Delta_T^{\mu\nu}$

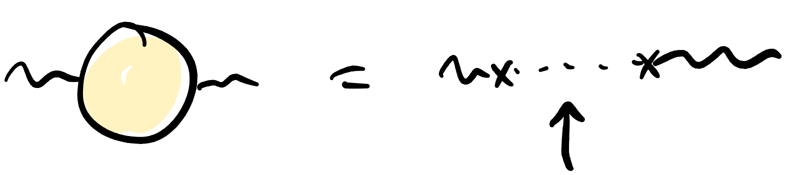
$$\Delta_T^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \quad \text{w/ } \Delta_T^2 = \Delta_T .$$

$\tilde{G}(q) =$ 

$$= \frac{-i\,\Delta_T}{q^2 - q^2\Pi(q^2)} .$$

Photon stays massless if $\lim_{q^2\to 0} q^2\Pi(q^2) = A_0 = 0.$

$A_0 \neq 0$ requires $\Pi(q^2) \sim \frac{A_0}{q^2}$



massless scalar : $\frac{1}{q^2}$

Anderson-Higgs mechanism

[why does $\Pi_2^{\mu\nu}$ = 

satisfy Ward $q_\mu \Pi_2^{\mu\nu}(q) = 0$

$$q_\mu \Pi_2^{\mu\nu}(q^2) = e^2 \int d^4p \, \mathrm{tr} \underbrace{\frac{1}{\not p+\not q-m}\not q\frac{1}{\not p-m}}\gamma^\nu$$

But:

$$\frac{1}{\not p+\not q-m}\not q\frac{1}{\not p-m} = \frac{1}{\not p-m} - \frac{1}{\not p+\not q-m}$$

$$= \int d^4p \left(f(p) - f(p+q)\right)$$

$$\stackrel{?}{=} \int dp\, f(p) - \int dp'\, f(p')$$

$p' \equiv p+q$

$$= 0 \checkmark$$

not true if $\int^{\Lambda} dp\, f(p)$ depends on Λ as $\Lambda \to \infty$.

<u>Hard cutoff breaks gauge invariance.</u>

$$G^{(2)}(q) \overset{q^2\to 0}{\sim} Z_\gamma \frac{-i\Delta_T}{q^2}$$

where $\underline{Z_\gamma = \frac{1}{1-\Pi(0)}} \simeq \underline{1 + \Pi_2(0) + \mathcal{O}(e^4)}.$

Claim: w/ a gauge-inv't regulator (w/ scale Λ),

$$\Pi_2(q^2) = \frac{\alpha_0}{4\pi}\left(-\frac{2}{3}\ln\Lambda^2 + 2D(q^2)\right)$$

finite as $\Lambda\to\infty$.

$$G^{(2)} \sim \frac{e_0^2\Delta_T}{q^2} \xrightarrow{\text{Renormalization}} Z_\gamma \frac{e_0^2\Delta_T}{q^2}$$

problem: $G^{(2)}$ depends on cutoff?

solution: Regard $e_0^2 = e_0^2(\Lambda)$ such that

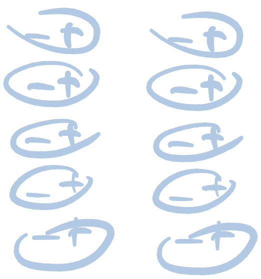
$e \equiv \sqrt{Z_\gamma}\, e_0$ is ind. of Λ.

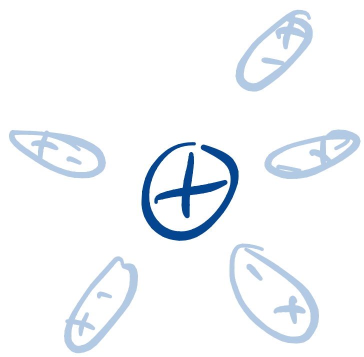
$$\Rightarrow \begin{cases} e_0^2(\Lambda) = e^2\left(1 + \frac{\alpha_0}{4\pi}\left(\frac{2}{3}\ln\Lambda^2 + 2D(0)\right)\right) + \mathcal{O}(\alpha^2) \\ m_0(\Lambda) = m + \mathcal{O}(\alpha_0) = m + \mathcal{O}(\alpha) \end{cases}$$

$$e^2 = e_0^2\left(1 - \frac{\alpha_0}{4\pi}\frac{2}{3}\ln\Lambda^2 + O(\alpha^2)\right)$$

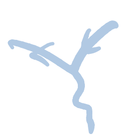
Real, measurable — fake — fake — fake

SCREENING: radiative corrections decrease the effective charge.





$$S_{e\mu\leftarrow e\mu} = \left(1 - \frac{\alpha_0}{4\pi}\ln\Lambda^2 + \frac{\alpha_0}{2\pi}A(m_0)\right)e_0^2\, L_\mu$$



$$\bar{u}(p')\left[\gamma^\mu\left(1 + \frac{\alpha_0}{4\pi}\ln\Lambda^2 + \frac{\alpha_0}{2\pi}(B+D)\right) + \frac{\alpha_0}{4\pi}\left(-\frac{2}{3}\ln\Lambda^2\right) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\frac{\alpha_0}{2\pi}C(q^2, m_0)\right] u(p)$$

A, B, C, D are finite.

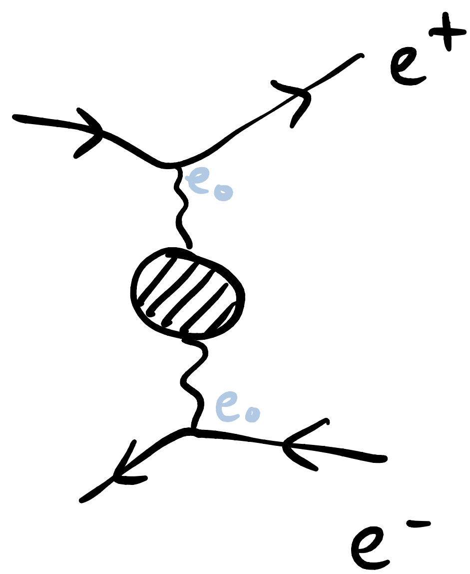
$$= e^2 L_\mu \, \bar{u}(p') \Big[\gamma^\mu \Big(1 + \frac{\alpha}{2\pi}(A+B+D) \Big) + i \frac{\sigma^{\mu\nu} q_\nu}{2m} \frac{\alpha}{2\pi} C \Big] u(p) + O(\alpha^2)$$

$e^2 = e_0^2(1 \ldots) + O(\alpha^2)$

$m = m_0 + O(\alpha)$

- cutoff independent!
- written in terms of physical e, m.

<u>eg:</u>



$$\propto e_0 \frac{1}{1-\Pi(0)} e_0 = e^2$$

what is $A+B+D$?

$$\mathcal{L}_{QED} = -\frac{1}{4e_{0(1)}^2} F_{\mu\nu} F^{\mu\nu} + \underbrace{i}_{(1+Z_\psi)} \bar{\Psi} \not{D} \psi - m_0 \bar{\Psi} \psi$$

method 1: $m^2\phi^2 \rightsquigarrow \underline{m_0^2(\Lambda)\phi^2}$

method 2: $m^2\phi \rightsquigarrow m^2\phi^2 + \delta_{m^2}(\Lambda)\phi^2$

$$= \left(m^2 + \delta_{m^2}(\Lambda)\right)\phi^2$$

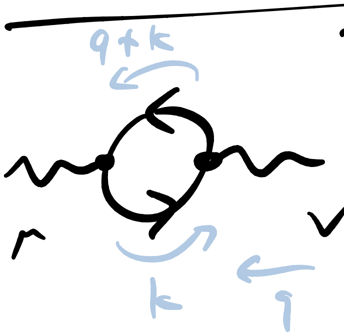
$$= m_0^2(\Lambda)$$

Renormalization conditions for QED:

① e propagator has a pole at $\not p = m$

② w/ residue 1

③ γ propagator $\propto \dfrac{e^2}{q^2}\Delta_T$.



fermion loop ↓

$$= -\int d^D k \; \mathrm{tr}\left((ie\gamma^\mu)\frac{i(\not k+m)}{k^2-m^2}(ie\gamma^\nu)\frac{i(\not q+\not k+m)}{(q+k)^2-m^2}\right)$$

① $\dfrac{1}{AB} = \displaystyle\int dx \frac{1}{(xA+(1-x)B)^2}$

② denom $= (\ell^2-\Delta)^2$

$$l = k + xq \qquad \Delta \doteq m^2 - x(1-x)q^2 .$$

$$\frac{N^{\mu\nu}}{4} = \underline{\underline{2 l^\mu l^\nu}} - \eta^{\mu\nu} l^2 - 2x(1-x)\, q^\mu q^\nu + \eta^{\mu\nu}\left(m^2 + x(1-x) q^2\right)$$

+ terms linear in l

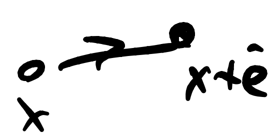
$$i\Pi_2^{\mu\nu}\Big|_{\substack{\text{hard}\\ \text{cutoff}}} \propto e^2 \int d^4 l_E \, \frac{l_E^2\, \eta^{\mu\nu}}{(l_E^2 + \Delta)^2}$$

$$\propto e^2 \Lambda^2 \eta^{\mu\nu} \quad \not\propto \Delta_T^{\mu\nu}$$

$$\Longrightarrow \delta m_\gamma^2 \propto \Lambda^2 .$$

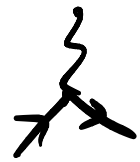
<u>on the lattice</u>: $\quad e^{i \int_x^{x+\hat{e}} A_\mu dx^\mu}$

$$= \underline{\underline{U(x, \hat{e})}}$$



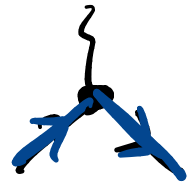
Fancier PV regulator :

$$\mathcal{L}_\psi(q,m) \equiv \bar\psi(\not\partial - q\not A)\psi - m\bar\psi\psi \rightsquigarrow \mathcal{L}_\psi(q,m) + \sum_{a=1}^{\dots} \mathcal{L}_{\psi_a}(\sqrt{c_a}, m_a)$$



$-iq\gamma^\mu$

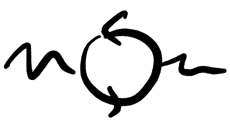
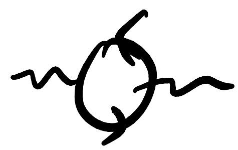
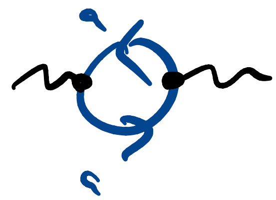
$q = 1$



$-i\sqrt{c_a}\,\gamma^\mu$

claim: The ψ_a do not change

 ,  .

But  $\rightsquigarrow$  $+ \sum_a$ 

$$\cancel{e_0^2}\; - \sum_a c_a \int d^Dk\, \mathrm{tr}\left(\overbrace{(i\gamma^\mu)\frac{i}{\not q + \not k - m_a}(i\gamma^\nu)\frac{i}{\not q - m_a}}^{\propto\, c_a} \right)$$

$$\sim \int^\Lambda d^4k \left(\underbrace{\frac{\sum_a c_a}{k^2}}_{\underbrace{\sum_a c_a \Lambda^2}_{\sum_a c_a \doteq -1.}} + \underbrace{\frac{\sum_a c_a m_a^2}{k^4}}_{\underbrace{\sum_a c_a m_a^2 \ln\Lambda^2}_{\sum_a c_a m_a^2 = -m^2}} + \underbrace{\dots}_{\text{finite}} \right)$$

work.

$$\rightsquigarrow \Pi_2(q^2) = \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\frac{M^2}{M^2 - x(1-x)q^2}$$

where $\ln M^2 \equiv -\sum_a c_a \ln M_a^2$

scale of UV ignorance.

In particular:

$$\Pi_2^{\mu\nu} = \underline{\underline{\Delta_T^{\mu\nu}}}\, q^2\, \Pi_2(q^2)$$

[Zee p202.]

Dim Reg: $\int d^4\ell \longrightarrow \int \frac{d^D\ell}{\bar\mu^{-\epsilon}}$ has same dims ($D = 4-\epsilon$)

$D \equiv 4-\epsilon$.

$\bar\mu$ will be the RG scale.

Breaks scale invariance.

Axioms of dim Reg:

① translations $\int d^Dp\, f(p+q) \overset{!}{=} \int d^Dp\, f(p)$.

$\longrightarrow \Pi_2^{\mu\nu}$ satisfies Ward.

② scaling $\int d^D p\, f(sp) = |s|^{-D} \int d^D p\, f(p)$

③ factorization $\int d^D p \int d^D q\; f(p) g(q)$

$$= \left(\int d^D p\, f(p)\right) \left(\int d^D q\, g(q)\right)$$

<u>axiom 3</u> :

$$\left(\frac{\pi}{a}\right)^{D/2} \overset{\downarrow}{=} \int d^D x\, e^{-a\vec{x}^2} = \Omega_{D-1} \int_0^\infty dx\, x^{D-1} e^{-ax^2}$$

$$= \underline{\underline{\Omega_{D-1}}} \cdot \frac{1}{2} a^{-D/2} \Gamma(D/2)$$

$\rightsquigarrow$

$$\Pi_2^{\mu\nu} = \Delta_T^{\mu\nu} q^2 \Pi_2(q^2) \qquad \left(\Delta = m^2 - x(1-x) q^2\right)$$

$$\Pi_2(q^2) \underset{\text{Peskin p.252}}{=} -\frac{8e^2}{(4\pi)^{D/2}} \int_0^1 dx\, x(1-x) \frac{\Gamma(2-D/2)}{\Delta^{2-D/2}} \bar{\mu}^{\epsilon}$$

$$\overset{D\to 4}{=} -\frac{e^2}{2\pi^2} \int dx\, x(1-x) \left(\frac{2}{\epsilon} - \underbrace{\log\left(\frac{\Delta}{\mu^2}\right)}_{\text{"finite"}} + \mathcal{O}(\epsilon)\right)$$

with $\mu^2 \equiv 4\pi e^{-\gamma_E} \bar{\mu}^2$

Renormalization condition: A] $Z_\gamma = 1 = \frac{1}{1-\Pi_2(0)}$

$$\Rightarrow \delta\Pi_2(q^2) = \Pi_2(q^2) - \Pi_2(0) \underset{\text{at } q^2=0}{\overset{!}{=}} 0$$

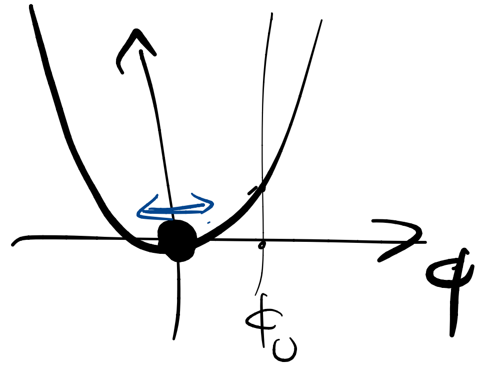
$$= \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x) \log\left(\frac{m^2 - x(1-x)q^2}{m^2}\right)$$

B] $\overline{MS}$ scheme : subtract the $\frac{1}{\epsilon}$ pole.

$$Z = \int D\phi \; e^{iS[\phi]}$$

$$\tilde{\phi} \equiv \tilde{\phi}(\phi)$$

eg 1: $V(\phi) = \frac{m^2}{2}\phi^2 + \frac{g\phi^4}{4!}$

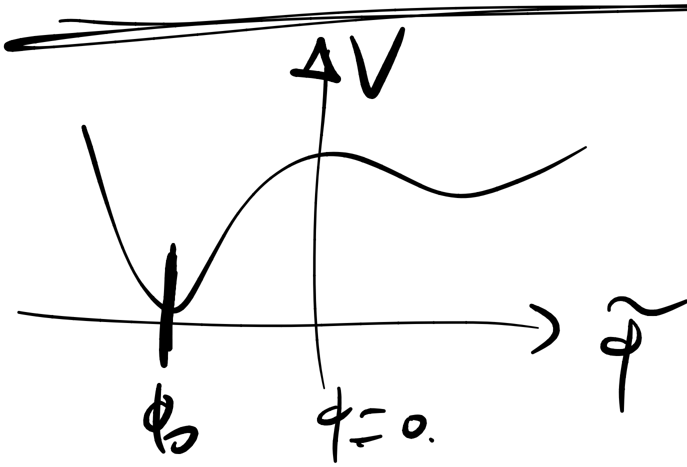


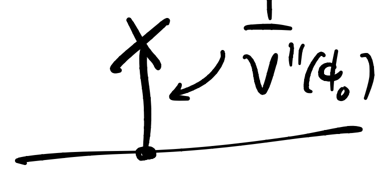
$$\longrightarrow \quad \phi(x) = \int \frac{d^3p}{\sqrt{2\omega_p}} \left(e^{ipx} a_p + h.c. \right)$$

$$\tilde{\phi} = \phi - \phi_0$$

$$\tilde{\phi} = \phi_0 + \int dp \left(e^{ipx} a + \dots \right)$$

eg 2:



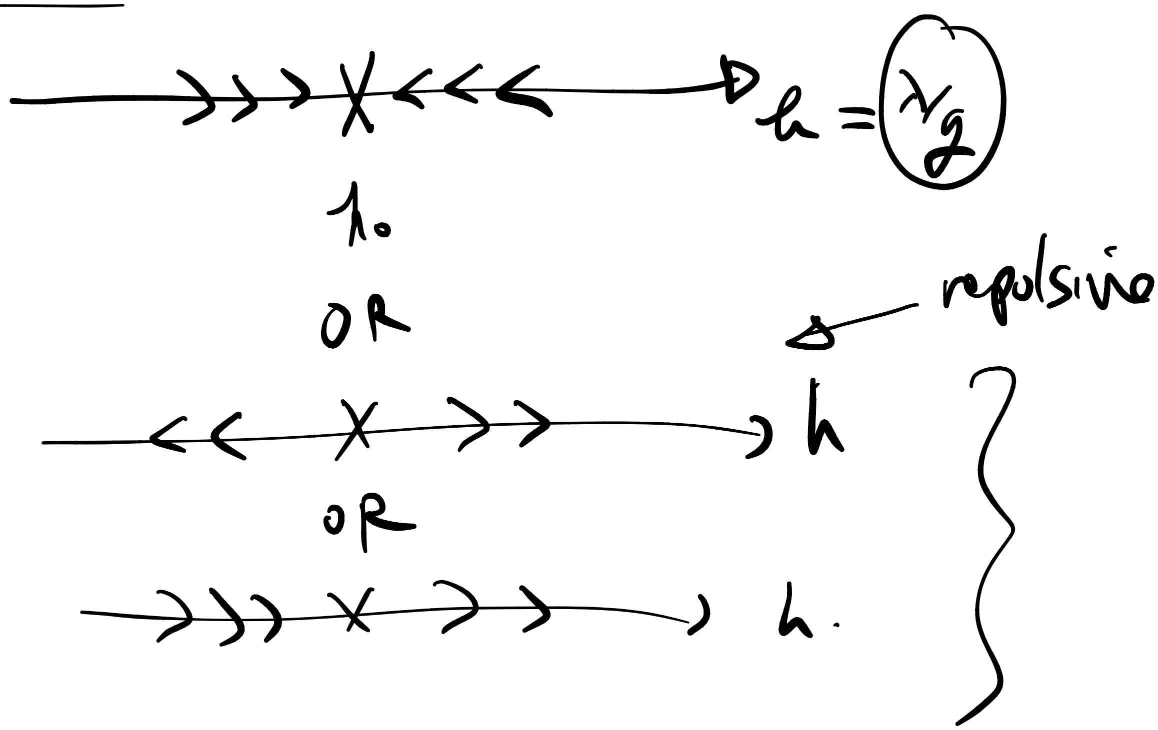


$$\frac{dk}{k} = d(\log_{10} k)$$

$$\text{total } \# = \int \frac{dk}{k} \, (\# \text{ per decade})$$

$$\mathcal{M}_{2\to 2}(s_0, t_0, u_0) \overset{!}{=} g_{phys}(s_0, t_0, u_0)$$

$$\underline{g_{phys} = g_0 + g_0^2 \ln s_0/\Lambda^2} \longrightarrow \underline{g_0^{(1)}} = g_{phys} - g_{phys}^2 \ln s_0/\Lambda^2$$



$$\begin{cases} \beta_\lambda = \Lambda \dfrac{d}{d\Lambda} \lambda_0(\Lambda) \\ \beta_g = \Lambda \dfrac{d}{d\Lambda} g_0(\Lambda) \end{cases}$$

$$\beta_{\lambda/g} = \Lambda \frac{d}{d\Lambda} \left(\frac{\lambda_0(\Lambda)}{g_0(\Lambda)} \right) =$$

$\parallel \leftarrow \parallel$