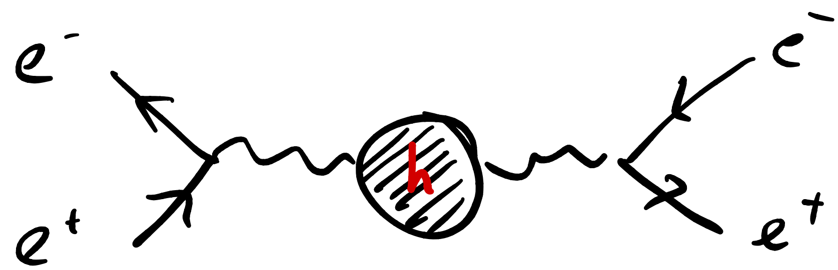
2.3 How to study hadrons with perturbative QCD

$$\sigma^{\,\text{hadrons}\ \cancel{\text{anything}} \xleftarrow{\gamma} e^+e^-} \overset{\text{optical thm}}{=} \frac{1}{2s}\,\mathrm{Im}\,\mathcal{M}_h\left(e^+e^- \xleftarrow{\gamma} e^+e^-\right)\Big|_{\text{forward}}$$

<u>forward:</u> $p_i = p_f$

$\alpha_i = \alpha_f$

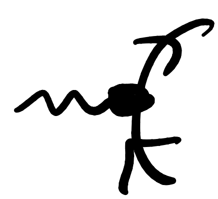


$= i\mathcal{M}_h$

<u>QCD:</u> $$\mathcal{L}_{QCD} = \sum_f \bar{q}_f \left(i\not{D} - m_f\right) q_f + \dots$$

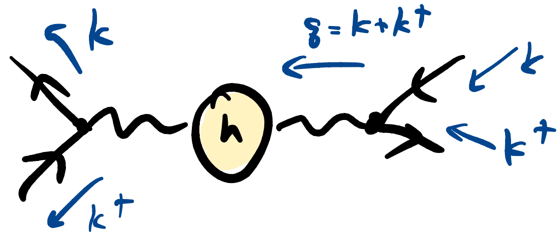
q_f : Dirac spinor.

$$(D_\mu)^{\alpha\beta} = \left(\partial_\mu - i Q_f A_\mu\right)\delta^{\alpha\beta} + \dots$$

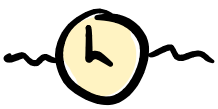


$$\bar{q}_f\, i\not{D}\, q_f = \bar{q}_{f\alpha}\,(i\not{D})^{\alpha\beta}\, q_{f\beta}$$

$= i Q_f \gamma^\mu$

$i\mathcal{M}_h =$ 

$$= (-ie)^2\ \bar{u}(k)\gamma_\mu v(k_+)\frac{-i}{s}\ \underline{\underline{i\Pi_h^{\mu\nu}(q)}}\frac{-i}{s} \times \bar{v}(k_+)\gamma_\nu u(k)$$



$$= i\Pi_h^{\mu\nu}(q) \overset{\text{Ward}}{=} i(q^2 g^{\mu\nu} - q^\mu q^\nu)\underline{\underline{\Pi_h(q^2)}}$$

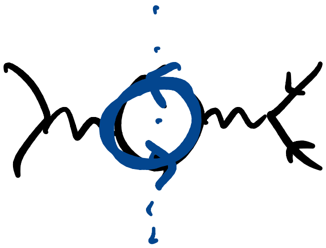
avg. optical thm over polarization

$$\sigma_{\substack{\text{hadrons} \leftarrow e^+e^- \\ \text{unpolarization}}} \overset{\downarrow}{=} \frac{1}{4}\sum_{\text{spins}} \frac{\text{Im}\,\mathcal{M}_h}{2s}$$

$$\left[\sum_{\text{spins}} \overbrace{\bar{u}\gamma \underbrace{v\ \bar{v}}_{\not{k}_+ - m}\gamma u}^{\not{k}+m} \overset{s \gg m_e}{=} \text{tr}(\not{k}_+\gamma^\mu \not{k}\gamma_\mu) = -8\,k\cdot k_+ = -4s\right]$$

$$= -\frac{4\pi\alpha}{s}\,\text{Im}\,\Pi_h(s). \leftarrow \left(\frac{1}{2}?\right)$$

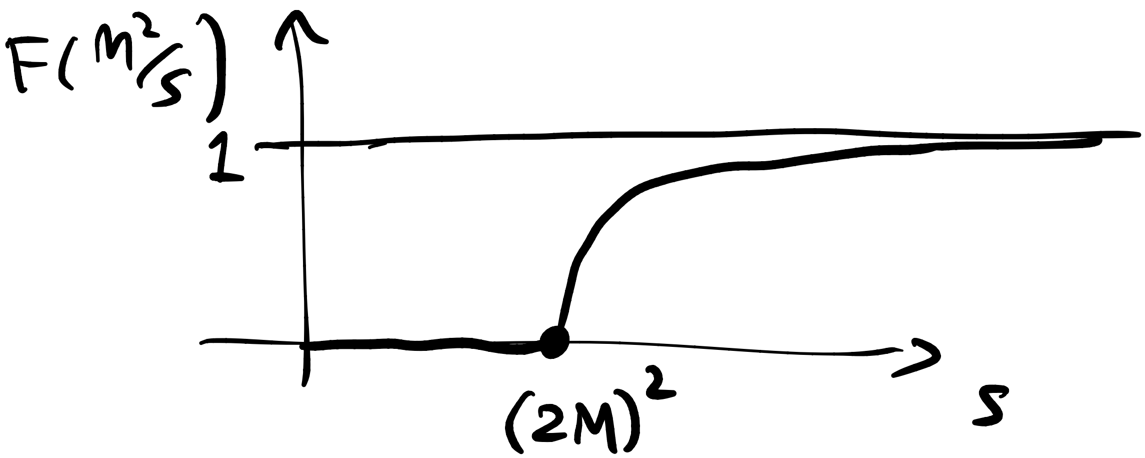
check: $\mathrm{Im}\,\Pi_L = -\frac{\alpha}{3} F\left(M^2/s\right)$ $\qquad$ L = lepton of mass M



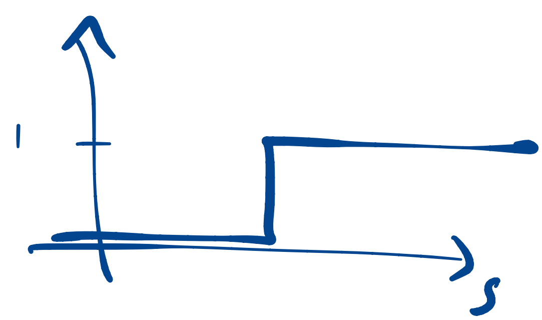
$$\sigma^{L^+L^- \leftarrow e^+e^-} = \underline{\underline{\frac{4\pi}{3}}}\,\frac{\alpha^2}{s} F\left(M^2/s\right)$$

$$= -\frac{4\pi\alpha}{s}\,\mathrm{Im}\,\Pi_L$$

$$F(x) = \begin{cases} 0 & x \geq 4 \\ \sqrt{1-4x}\,(1+2x) \simeq 1 + \mathcal{O}(x) & x \leq 4 \end{cases}$$

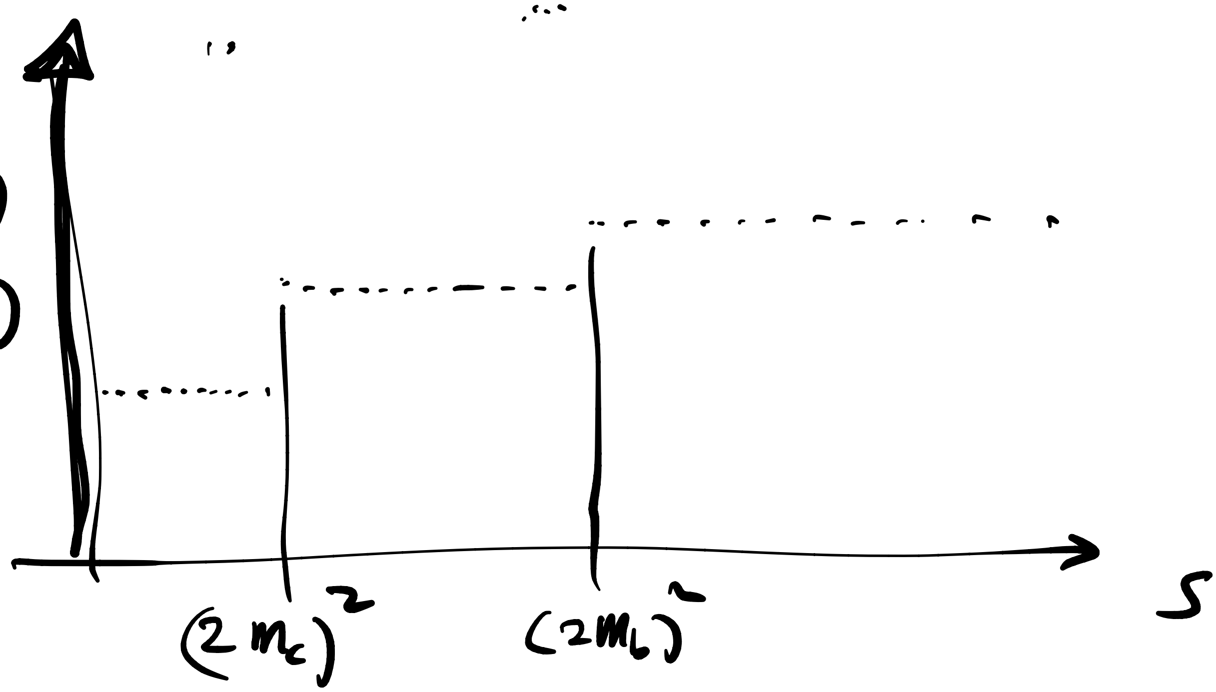


$\sim$



$$\sigma_0^{\text{quarks} \xleftarrow{\gamma} e^+e^-} = 3 \sum_{\substack{\text{flavors} \\ f}} Q_f^2\, \frac{4\pi}{3}\frac{\alpha^2}{s} F\left(m_f^2/s\right)$$

$$R \equiv \frac{\sigma(\text{hadrons} \xleftarrow{\gamma} e^+e^-)}{\sigma(\mu^+\mu^- \xleftarrow{\gamma} e^+e^-)}$$

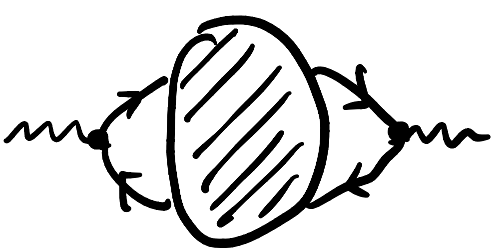


$m_c \cong 1.3$ GeV $\qquad m_b \cong 4.5$ GeV

<u>Q</u>: why does tree-level QCD do well?

$$i\Pi_h^{\mu\nu}(q) = -e^2 \int d^4x\, e^{-iqx} \langle \Omega | T J^\mu(x) J^\nu(0) | \Omega \rangle$$

$q^2 \gg m_e^2$ means x is small?



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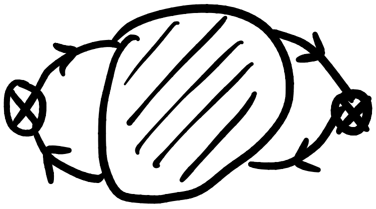
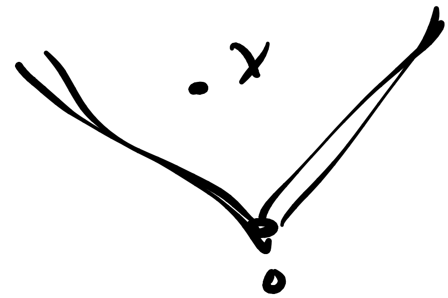


quark part of EM current:

$$J^\mu(x) = \sum_f Q_f \bar{q}_f(x) \gamma^\mu q_f(x)$$

In $\sigma(\ldots \leftarrow e^+e^-)$, we're interested in timelike q^μ. $\underline{\underline{q^2>0}}$.

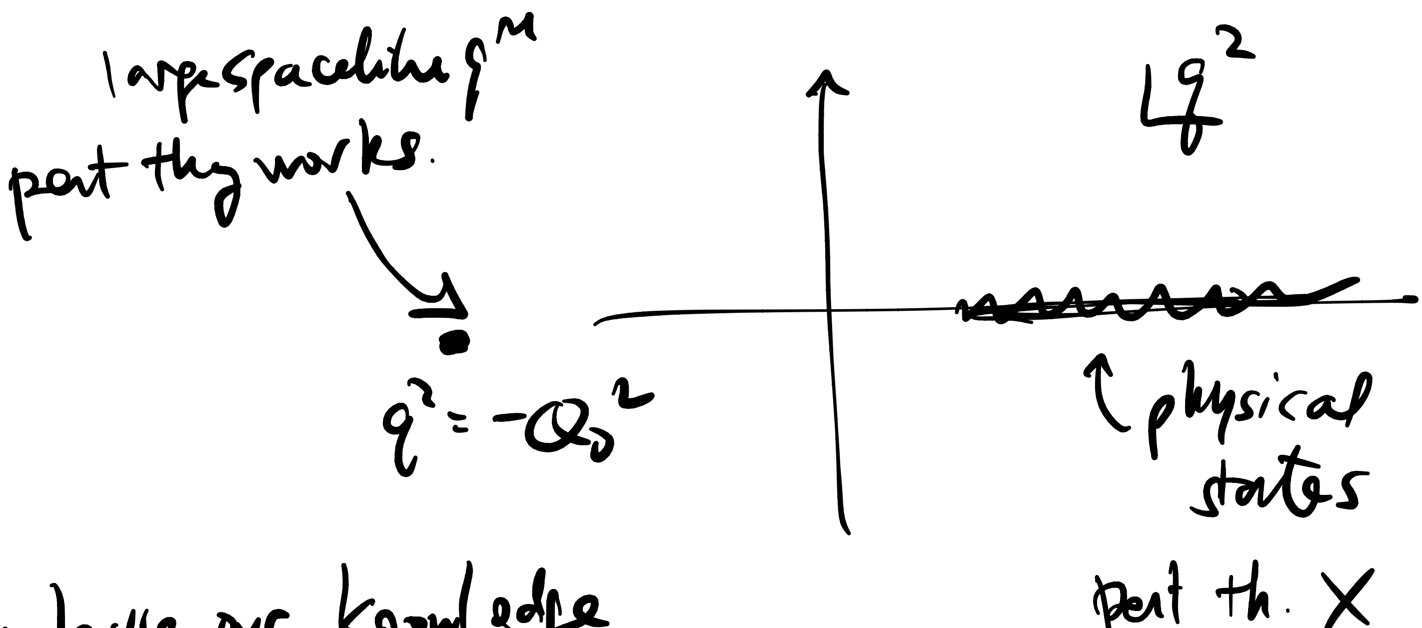
$\Rightarrow$ dominant contribution of dx are timelike x^μ.



$$J^\mu(x) \; {}_\wedge \; J^\nu(0)$$

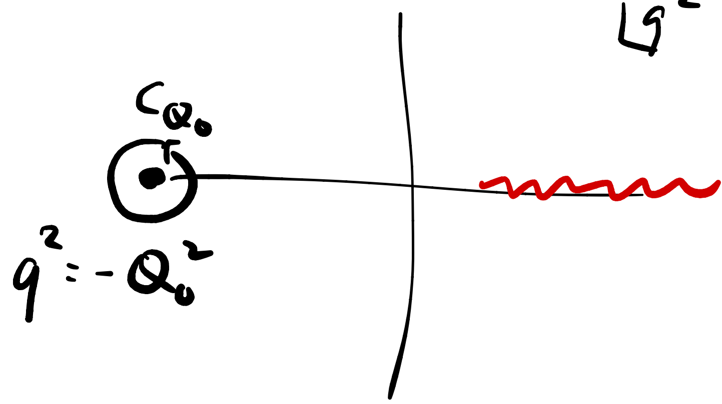
$$\mathbb{1} = \sum_n |n\rangle\langle n|$$

$\uparrow$ physical states = hadrons.



<u>Q</u>: how to use our knowledge of $\Pi(q^2 = -Q_0^2)$ to learn about $\Pi(q^2 \gg 0)$?

$$I_n \equiv -4\pi\alpha \oint_{C_{Q_0}} \frac{dq^2}{2\pi i} \frac{\Pi_h(q^2)}{(q^2+Q_0^2)^{n+1}}$$



$$= -\frac{4\pi\alpha}{n!} (\partial_{q^2})^n \Pi_h \Big|_{q^2=-Q_0^2} .$$ $\Leftarrow$ can be calculated in pert. thy.

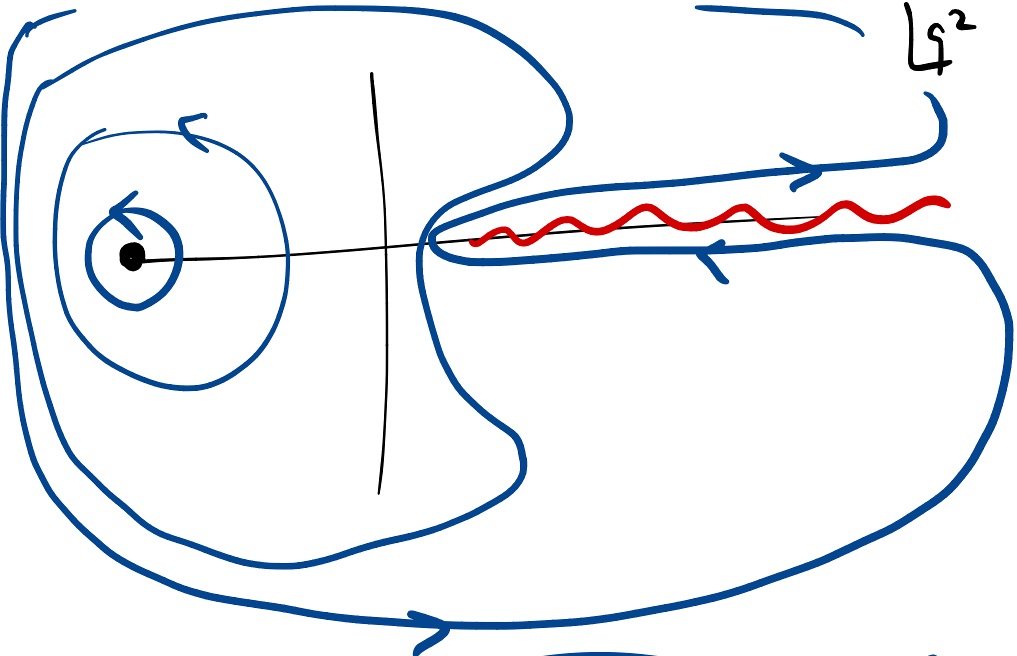
Principle of locality:

singularities of $\Pi_h(q^2)$ only arise for physical reasons — on-shell intermediate states.

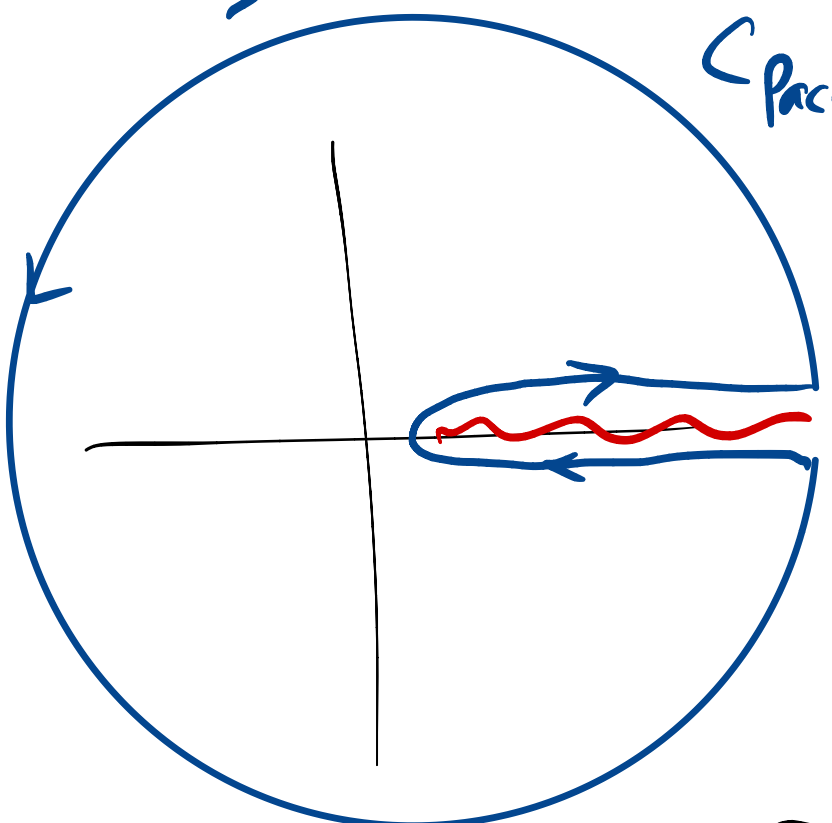
$\Rightarrow$ can deform the contour w/o changing I_n (away from $\mathbb{R}_+$.)

Spectral Rep. for vectors $\Longrightarrow$ $\left(\begin{array}{c}\text{like} \\ D(k) \leq \frac{1}{k^2}\end{array}\right)$

$$\Pi_h(q^2) \overset{|q| \gg \ldots}{\leq} \log(q^2) .$$



C_{Pacman}



$$\Pi(x+i\epsilon) - \Pi(x-i\epsilon) \equiv \operatorname{Disc}\Pi(x)$$

$$I_n = -4\pi\alpha \oint_{\text{Pacman}} \frac{dq^2}{2\pi i} \frac{\Pi_h(q^2)}{(q^2+Q_0^2)^{n+1}}$$

if $n \geq 1$

$$= -4\pi\alpha \int \quad \cdots = -4\pi\alpha \int_{s_{\text{threshold}}}^{\infty} \frac{ds}{2\pi i} \frac{\operatorname{Disc}\Pi_h(q^2)}{(q^2+Q_0^2)^{n+1}}$$

$$\mathrm{Disc}\,\Pi = 2i\,\mathrm{Im}\,\Pi$$

$$\Rightarrow I_n = -4\pi\alpha \int_{s_{thr.}}^{\infty} \frac{ds}{2\pi i} \frac{2i\,\mathrm{Im}\,\Pi}{(s+Q_0^2)^{n+1}}$$

optical thm

$$= \frac{1}{\pi} \int_{s_{thresh}}^{\infty} ds \frac{s}{(s+Q_0^2)^{n+1}} \sigma^{\text{hadrons} \leftarrow e^+e^-}(s)$$

Can calculate in pert. th.!

Can measure!

[ITEP sum rule.]

$$\overset{?}{\Rightarrow} \sigma(s) \overset{?}{=} \sum_n (f(s)) I_n$$

no. $\rightarrow$ can't see narrow resonances in fixed-order pert. thy.

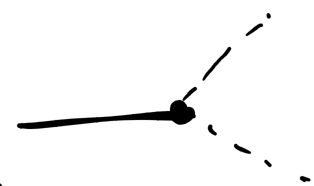
Peskin §18.4

Schwartz §24.

3 Parable on Integrating out heavy dofs

$$S[Q,q] = \int dt \left[\frac{1}{2}(\dot{q}^2 + \omega_0^2 q^2) + \frac{1}{2}(\dot{Q}^2 + \Omega^2 Q^2) + g\, Q q^2 \right]$$

$$\equiv S_{\omega_0}[q] + S_{\Omega}[Q] + S_{int}[Q,q].$$



$$Z_{eucl.} = \int [dQ\,dq]\, e^{-S[Q,q]}$$

Let's Do $\int [dQ]$ first $\equiv$ "integrating out Q"

$$e^{-S_{eff}[q]} \equiv \int [dQ]\, e^{-S[Q,q]}$$

$$\left(Z = \int [dq]\, e^{-S_{eff}[q]} \right)$$

$$= e^{-S_{\omega_0}[q]} \left\langle e^{-S_{int}[Q,q]} \right\rangle_Q$$

$$\langle \ldots \rangle_Q \equiv \int [dQ]\, e^{-S_\Omega[Q]} \ldots \quad \text{gaussian.}$$

$$\langle e^{-S_{int}[Q,q]} \rangle_Q = \int [dQ]\, e^{-S_\Omega[Q] - \int ds\, J(s) Q(s)}$$

$$J(s) = g q^2(s) \qquad = \mathcal{N} e^{\frac{1}{4}\int ds\, dt\, J(s) G(s,t) J(t)}$$

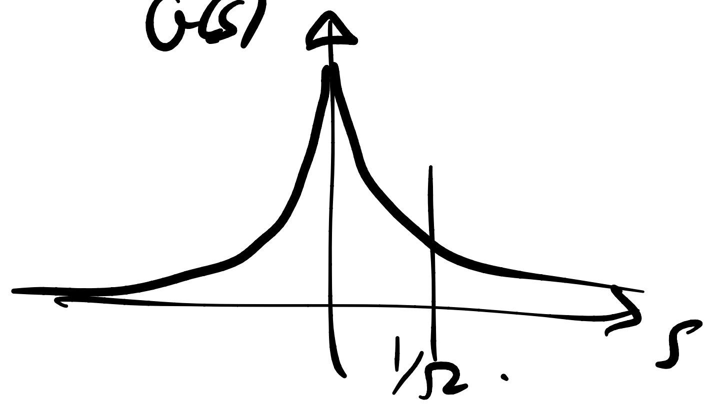
w/ $S_\Omega[Q] = \int dt\, ds\; Q(s)\, G^{-1}(s,t)\, Q(t)$

i.e. $(-\partial_s^2 + \Omega^2)\, G(s,t) = \delta(s-t).$

$G(s,t) = G(s-t).$

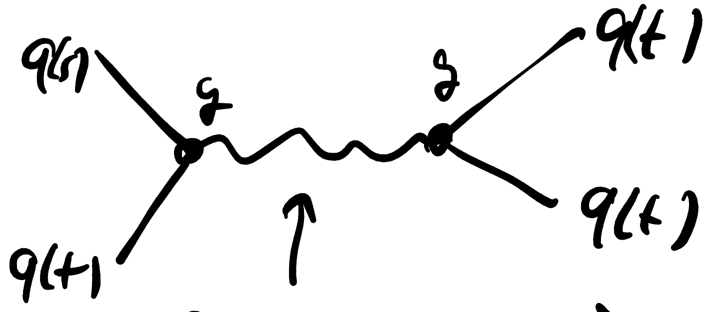
$G(s) = \int d\omega\, e^{-i\omega s} G_\omega \Rightarrow G_\omega = \dfrac{1}{\omega^2 + \Omega^2}.$

$$G(s) = \int d\omega \frac{e^{-i\omega s}}{\omega^2+\Omega^2} \overset{\text{Cauchy}}{=} \frac{e^{-|s|\Omega}}{2\Omega}.$$



$$e^{-S_{eff}[q]} = e^{-S_{\omega_0}[q] - \int dt\, ds\, \frac{g^2}{2} q(s)^2 G(s,t) q(t)^2}$$

$$S_{eff}[q] = S_{\omega_0}[q] + \int dt\, ds\, \frac{g^2}{2} q(s)^2 G(s,t) q(t)^2$$



$$G(s,t) = \langle Q(s) Q(t) \rangle_Q$$

$\sim g^4$.

But: S_{eff} is non-local. $\int ds\, dt$

BAD: $0 = \frac{\delta S}{\delta q(t)} = -\ddot{q} + \omega_0^2 q + \int ds \cdots$

eom depend on history!

Suppose: we're interested only in timescales $\Delta t \gg 1/\Omega$.

e.g. $\sim \frac{1}{\omega_0}$

assume $\omega_0 \ll \Omega$.

$$G(s) = \int d\omega \, \frac{e^{-i\omega s}}{\omega^2 + \Omega^2}$$

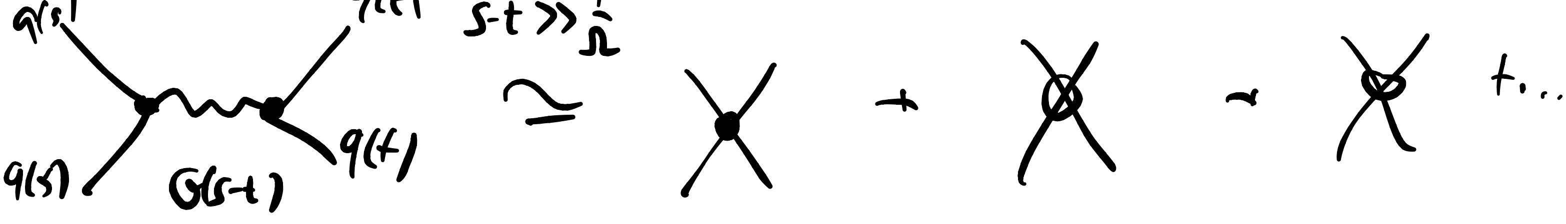
$$= \int d\omega \, \frac{e^{-i\omega s}}{\Omega^2} \underbrace{\frac{1}{1 + \omega^2/\Omega^2}}_{\overset{s \gg 1/\Omega}{=} \sum_{n=0}^{\infty} (-1)^n \left(\frac{\omega^2}{\Omega^2}\right)^n}$$

$$= \int d\omega \, \frac{e^{-i\omega s}}{\Omega^2} \left(1 - \frac{\omega^2}{\Omega^2} + \dots \right)$$

$$= \frac{1}{\Omega^2} \delta(s) + \frac{1}{\Omega^4} \partial_s^2 \delta(s) + \dots \quad \mathcal{O}\left(\frac{\partial_s^4}{\Omega^6}\right)$$

$$S_{\text{eff}}[q] = S_{\omega_0}[q] + \int dt \, \frac{g^2}{2\Omega^2} q(t)^4$$

$$+ \int dt \, \frac{g^2}{2\Omega^4} \dot{q}^2 q^2 + \mathcal{O}\left(\frac{\partial_t^4}{\Omega^6}\right)$$



$$\frac{q^4}{\Omega^2} \qquad \frac{\dot{q}^2 q^2}{\Omega^4} \qquad \ldots$$

for an experiment w/ precision Δ

at energy ω

keep up to nth term

$$\left(\frac{\omega}{\Omega}\right)^{2n} \sim \Delta$$

$$S_{eff}[q] = \ldots \sum_n c_n \left(\partial_t^n q\right)^2 q^2$$

w/ $c_n \sim \frac{1}{\Omega^{2n}}$. $\underline{[c_n] < 0}$.

$$\Rightarrow S_{qq\leftarrow qq} \sim E^{4n-2} c_n^2$$

violates unitarity when $\underline{E > \Omega}$.

This is the basis of <u>Effective Field Theory</u> (EFT):

<u>coarse graining</u>: focus the low-E dofs (q)

actively ignore high-E dofs (Q)

except to the extent that they affect the low-E modes

$$S_{eff}[q].$$

any weakly-coupled EFT = collection of (coupled) oscillators

w/ $\Omega = \sqrt{\vec{k}_h^2 + m^2}$

$\omega_0 = \sqrt{\vec{k}_\ell^2 + m^2}$

k

k_h

k_ℓ

do the integrals in this order.

<u>Result</u>: generate all terms in $S_{eff}[q]$ consistent w/ symmetries.

Asymmetry: If we did $\int[dq]$ first, we can't get a local action.