4.3 Interlude on Differential Forms

Given a smooth manifold X w/ (local) coords x^μ.

A p-form on X is

$$\mathbf{A} = \frac{1}{p!} \underbrace{A_{m_1 \ldots m_p}}_{\text{completely Antisymmetric}} \underline{dx^{m_1} \wedge \ldots \wedge dx^{m_p}}$$

$$A_{m_1 m_2 \ldots} = - A_{m_2 m_1} .$$

if $\tilde{x}^\mu = \tilde{x}^\mu(x)$ then

$$d\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu .$$

$$\left.\begin{array}{l} dx \wedge dy = -dy \wedge dx \\ dx \wedge dx = 0 . \end{array}\right\} \text{(like fermionic fields)}$$

$D=2$.

$$d\tilde{x} \wedge d\tilde{y} = J(x,y)\, dx \wedge dy$$

$$J(x,y) = \det \frac{\partial(\tilde{x},\tilde{y})}{\partial(x,y)}$$

The point of a p-form on X is to be integrated over a p-dim'l submfld of X.

eg: $A = A_\mu dx^\mu \quad (p=1)$

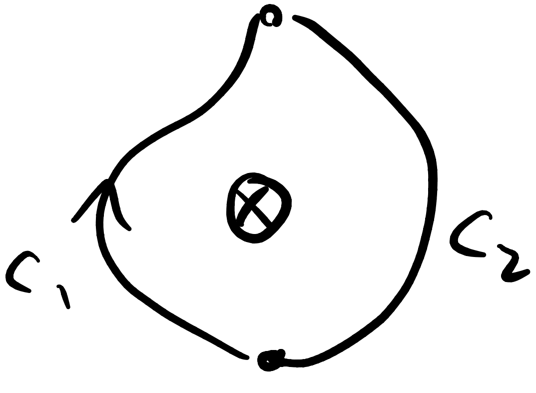
$$\int_C A \equiv \int_C dx^\mu A_\mu(x) = \int ds \frac{dx^\mu}{ds} A_\mu(x(s))$$

$x^\mu(s)$ is a parametrization of C.

is geometric ≡ indep. of parametrization of C

Phase acquired by a unit charge particle moving along C in a background EM field A:

$$e^{i\int_C A}.$$



eg 2: $p=2$ $F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \underline{\underline{\wedge}}\, dx^\nu$.

- **wedgeproduct:** of a p-form A and a q-form B is a (p+q) form :

$$A\wedge B = \underline{\underline{A_{\mu_1\ldots\mu_p}}}\,\underline{\underline{B_{\mu_{p+1}\ldots\mu_{p+q}}}}\; dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_{p+q}} .$$

$$A_p \wedge B_q = (-1)^{pq}\, B_q \wedge A_p$$

Space of smooth p-forms on $X \equiv \Omega^p(X)$

is a **vector space** : $\left(\begin{array}{c} a_1 A_1 + a_2 A_2 \in \Omega^p(X) \\ \text{if } A_{1,2} \in \Omega^p(X) \end{array} \right)$

- **Exterior derivative** $d : \Omega^p(X) \to \Omega^{p+1}(X)$

$$A \mapsto dA$$

$$= dx^\nu \wedge \frac{\partial}{\partial x^\nu} A$$

$$dA \equiv \frac{1}{p!}\, \partial_{\mu_1} A_{\mu_2\ldots\mu_{p+1}}\, dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_{p+1}} .$$

claim: $d^2 = 0$

pf: $[\partial_\mu, \partial_\nu] = 0$. (on smooth functions)

Stokes Thm:
$$\boxed{\int_{D_p} d\alpha_{p-1} = \int_{\partial D_p} \alpha_{p-1}}$$

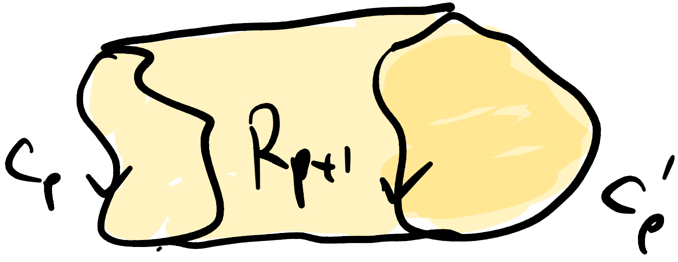
- $\Omega^{p>\dim(X)}(X) = 0$.

A form ω_p is **closed** if $d\omega_p = 0$.

If $d\omega_p = 0$ then $\int_{C_p} \omega_p$ is "topological"

in the sense:

$$\int_{C_p} \omega_p - \int_{C_p'} \omega_p = \int_{C_p - C_p'} \omega_p = \int_{\partial R_{p+1}} \omega_p \overset{\text{Stokes}}{=} \underbrace{\int_{R_{p+1}} d\omega_p}_{=0}$$



related by continuous deformation

A form ω_p is <u>exact</u> if $\omega_p = d\alpha_{p-1}$

then $\int_{C_p} \omega_p \overset{\text{stokes}}{=} \int_{\partial C_p} \alpha_{p-1} = 0$ if $\partial C_p = 0$.

<u>Note</u>: $d^2 = 0 \Rightarrow$

exact $\Rightarrow$ closed.

$\omega = d\alpha \Rightarrow d\omega = d^2\alpha = 0$.

But <u>closed $\not\Rightarrow$ exact.</u>

<u>Illustrations</u>: $X = \mathbb{R}^3$

$\Omega^0(\mathbb{R}^3)$ = functions = $\Omega^3(\mathbb{R}^3)$

$= \{ f(x,y,z) \, \underline{dx \wedge dy \wedge dz} \}$

$\Omega^1(\mathbb{R}^3)$ = vector fields = $\Omega^2(\mathbb{R}^3)$

$= \{ f_i \, dx^i \}$ $\{ f_i \, \epsilon_{ijk} \, dx^j \wedge dx^k \}$

on Ω^0 : $df = \partial_i f \, dx^i$ **GRAD**

on Ω^1 : $d(f_i dx^i) = (\partial_y f_z - \partial_z f_y) dy \wedge dz$

$$+ (\partial_x f_y - \partial_y f_x) dx \wedge dy$$

$$+ (\partial_z f_x - \underline{\partial_x} f_z) dz \wedge dx$$

$$= \frac{1}{3!} \epsilon_{ijk} \partial_i f_j \epsilon_{klm} dx^l \wedge dx^m$$

CURL

on Ω^2 : $f_x \, dy \wedge dz$ + cyclic

$$\begin{tikzcd} x \arrow[r] & y \arrow[dl] \\ z \arrow[u] & \end{tikzcd}$$

$$\xmapsto{d} \partial_i f_i \, dx \wedge dy \wedge dz$$ **DIV**

on Ω^3 : $d = 0$.

$d^2=0$ means $\underline{\text{exact} \Rightarrow \text{closed}}$.

$$\mathrm{Im}\, d : \Omega^{p-1} \to \Omega^p \quad \leftarrow \text{exact}$$

$$\Longleftrightarrow \quad \subset \mathrm{Ker}\, d : \Omega^p \to \Omega^{p+1}$$

$\leftarrow$ closed

$$\cdots \to \underline{\underline{\Omega^{p-1}}} \xrightarrow{d} \underline{\underline{\Omega^{p}}} \xrightarrow{d} \Omega^{p+1} \to \cdots$$

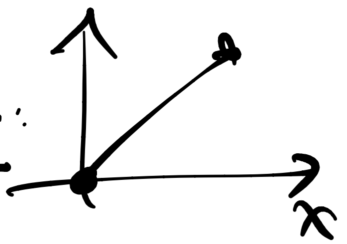
$$H^p(X) = \frac{\mathrm{Ker}(d) < \Omega^p}{\mathrm{Im}(d) < \Omega^p} = \frac{\text{closed}}{\text{exact}}$$

de Rham cohomology. top. invt of X.

$b^p \equiv \dim H^p(x)$ betti #s.

<u>eg</u>: $\underline{X = S^1}$. $\underline{x \simeq x+2\pi}$.

<u>not periodic</u>:



$\Omega^0(S^1) =$ smooth <u>periodic</u> fn of x.

$$\Omega^1(S^1): \quad \underbrace{A_1}_{\text{smooth periodic f'n.}}(x)\,dx$$

$$0 \longrightarrow \underline{\underline{\Omega^0(S^1)}} \xrightarrow{d} \underline{\underline{\Omega^1(S^1)}} \longrightarrow 0$$

$$d A_0(x) = \underline{\underline{A_0'(x)\,dx}}$$

$\operatorname{Ker} d \subset \Omega^0(S^1)$ = constant f'ns

$A_0'(x) = 0$

$H^0(S^1)$ = constant f'ns $\Rightarrow b^0(S^1) = 1$

which forms $A_1(x)dx = A_0'(x)dx$

$$A_0 = \int A_1$$

The only A_1 that we can't get this way is $A_1 = 1$.

$x \xmapsto{d} dx$ but x is $\underline{\underline{\text{not}}}$ periodic!!

$\Rightarrow \Omega^1(x) = \{\text{const}\cdot dx\}$

conclusion: $b^0(S^1) = b^1(S^1) = 1$.

Context: fluid dynamics: given $\vec{u}$ when is it $\vec{u} = \vec{\nabla}\phi$?

electrostatics: $\vec{E} = -\vec{\nabla}\phi$

Suppose we have a metric on X

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu .$$

Hodge star: $* : \Omega^p \longrightarrow \Omega^{D-p}$

($D \equiv \dim X$)

$$\left(* A^{(p)}\right)_{\mu_1 \dots \mu_{D-p}} \equiv \frac{\sqrt{g}}{p!}\, \epsilon_{\mu_1 \dots \mu_D} \left(A^{(p)}\right)^{\mu_{D-p+1} \dots \mu_D}$$

$\sqrt{g} \equiv \sqrt{|\det g|}$. indices raised by $g^{\mu\nu}$.

Illustration: EM field:

$$\underline{\underline{F = dA}}.$$

$$e^{i\oint_{C_1-C_2} A} \overset{\text{stokes}}{=} e^{i\int_R F}$$

$$= e^{i\underbrace{\int B}_{\text{magnetic flux}}}.$$

$$F = dA = E_i\, dx^i \wedge dt + B_i\, dx^j \wedge dx^k\, \epsilon_{ijk}/2$$

$$*F = -B_i\, dx^i \wedge dt + E_i\, dx^j \wedge dx^k\, \epsilon_{ijk}/2$$

$$= \left(\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}\right) dx^\mu dx^\nu.$$

Maxwell's eqns: $dF = 0$. $d*F = 0$

w/o matter

w/ matter: $$\left\{\begin{array}{l} dF = *j_m \\ d*F = *j_e \end{array}\right.$$

inv't under
$F \leftrightarrow *F$
$j_m \leftrightarrow j_e$.

EM duality.

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w/o matter: $dF = 0$ is solved by $F = dA$. (Bianchi id)

$$0 = d*F \propto \frac{\delta S_{Max}}{\delta A} \quad \text{w/}$$

$$S_{Max}[A] = -\frac{1}{2e^2} \int_X \underbrace{\underbrace{F}_{2} \wedge \underbrace{* \underset{2}{F}}_{D-2}}_{D\text{-form}}$$

$$= -\frac{1}{4e^2} \int d^D x \sqrt{g}\, F_{\mu\nu} F^{\mu\nu}$$

Max eqns in vac: F, $*F$ represent elements of $H^2(X)$

charges.

Simplest case: point charge at rest at $\vec{0}$.

$$F = q\,\frac{dr\wedge dt}{r^2} = -q\,d\left(\frac{dt}{r}\right)$$ is exact.

$$\frac{dt}{r} \in \Omega^1(\underbrace{\mathbb{R}^4 \setminus \mathbb{R}_t}_{\equiv M}) \quad \uparrow_{\text{the origin}}.$$

But: $$*F = q\,\frac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{r}$$

$$\left(\begin{aligned} dr &= \frac{x^i dx^i}{r} \\ *dx\wedge dt &= dy\wedge dz \cdots \end{aligned} \right)$$

$\in H^2(M)$ nontrivial.

ie $*F \neq d\alpha$.

$$\int_{S^2} *F = 4\pi q. \neq 0$$

S^2 ↑ unit sphere about $\vec{0}$

Abelian p-form gauge fields in D dims:

$F_2 = dA_1$ inv't under $A_1 \to A_1 + d\lambda_0$

(since $d^2 = 0$.)

generalization: $F_{p+1} = dA_p$

$\delta A_p = d\lambda_{p-1}$ is a gauge redundancy.

($\delta F = 0$ by $d^2 = 0$)

$$S[A] = -\frac{1}{2g^2}\int F_{p+1} \wedge * F_{p+1}$$

$$\overset{?}{=} -\frac{1}{g^2(p+1)!}\int \sqrt{g}\, F_{\mu_1 \cdots \mu_{p+1}} F^{\mu_1 \cdots \mu_{p+1}}$$

eg: $p = 0$: $\mathcal{L} = -\frac{1}{2\underline{\underline{g}}^2}(\partial_\mu \phi)(\partial^\mu \phi)$. <u>spin 0</u>

(w/ $\underline{\phi \cong \phi + 2\pi}$)

$0 = \frac{\delta S}{\delta A} \propto d * F$.

Couple to charged matter:

$$\Delta S = e \int_{X_p} A_p$$

$\uparrow$ X_p ← worldvolume of a charged brane.

EM duality:

$$dA_p = * d \underset{\substack{\uparrow \\ \text{dual potential.}}}{A^\vee_{D-p-2}}$$

eg: $p=1$, $D=4$

$$F = dA$$
$$= * d\underline{\underline{A^\vee}}.$$

Path Integral derivation of duality:

$$Z = \int [dA]\, e^{-\frac{1}{2g^2}\int dA \wedge * dA}$$

$$= \int \frac{[dA\ \overset{\text{D-p-2 form potential}}{dA^\vee}\ \overset{\text{p+1 form potential}}{dB}]}{*}\ e^{i\int B \wedge dA^\vee}\ e^{-\frac{1}{2g^2}\int (F-B)\wedge *(F-B)}.$$

- this model has a Redundancy :

$$* : \begin{cases} A \to A + \Lambda \\ B \to B + d\Lambda \\ A^\vee \to A^\vee \end{cases} \qquad \Lambda \in \Omega^p(X)$$

($F - B$ is invt under $*$)

$$\delta \int B \wedge dA^\vee = \int d\Lambda \wedge dA^\vee$$

$$\overset{IBP}{=} -\int \Lambda \wedge d^2 A^\vee$$

$$= 0.$$

- $\int [dA^\vee] \, e^{-i \int dB \wedge A^\vee}$

$$\overset{H^{p+1}(X)=0}{=} \delta[dB]$$

$$dB = 0 \xRightarrow{H^{p+1}(X)=0} B = d\Lambda$$

$\Longrightarrow$ can choose a gauge for $*$ to set $B = 0$.

$\rightsquigarrow$ Maxwell theory.

instead: choose Λ to set $A = 0$.

$$\rightsquigarrow \int [dB\, dA^\vee]\, e^{-\frac{1}{2g^2}\int B\wedge *B + i\int B\wedge dA^\vee}$$

$\int dB$ is gaussian!

$$= \int [dA^\vee]\, e^{-\frac{g^2}{2}\int dA^\vee\wedge * dA^\vee}$$

Summary:

$$-\frac{1}{2}\frac{1}{g^2}\int dA_p \wedge * dA_p \overset{\text{dual}}{\simeq} -\frac{1}{2} g^2 \int d\check{A}_{D-p-2} \wedge * d\tilde{A}_{D-p-2}$$

$$g \longleftrightarrow 1/g$$

$$A_p \leftrightarrow \tilde{A}_{D-p-2}.$$

Special case $D=2,\ p=0$:

$$\frac{1}{R^2}(\partial\phi)^2 \simeq R^2(\partial\tilde{\phi})^2$$

"T-duality".

Poincaré duality :

$$* : \Omega^p \to \Omega^{D-p}$$

in fact $* : H^p \to H^{D-p}$

is an isomorphism if X has a volume form.

$$b^p = b^{D-p} .$$

Consider : a QFT μ

fields = maps : spacetime $\longrightarrow X$

non-linear sigma model. (NLSM)

Coordinate differentials $dX^\mu = \psi^\mu$

fermi fields

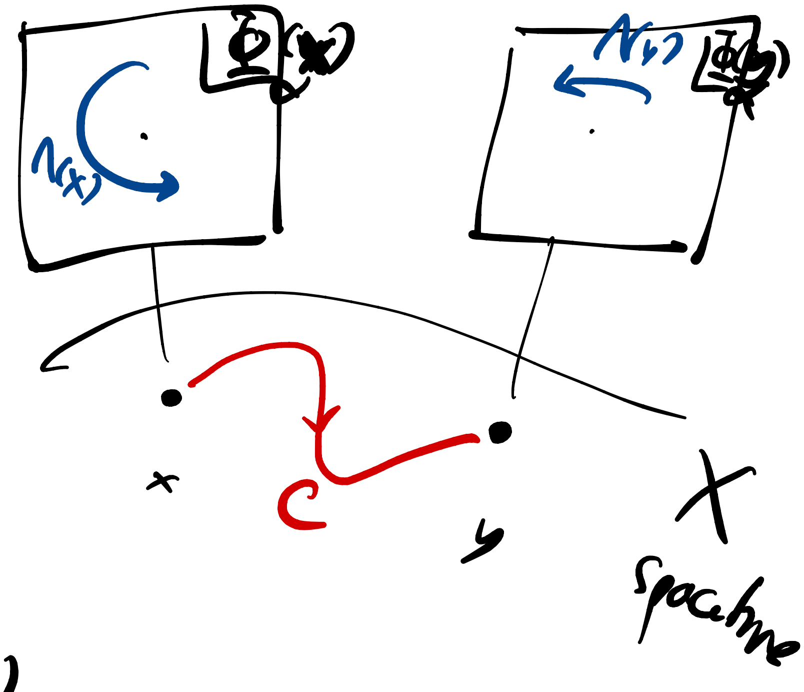
supersymmetry.

groundstates of susy NLSM on X

$$\longleftrightarrow H^\bullet(X) .$$

[Witten]

4.4 Gauge fields as connections



over each point in spacetime, $\mathbb{C}^N$
a vector space w/ coord $\Phi_\alpha(x)$ $\alpha=1..N$

w/ an action of SU(N):

$$\Phi_\alpha(x) \to \underline{\underline{\Lambda_{\alpha\beta}(x)}}\,\Phi_\beta(x)$$

change of basis of $\mathbb{C}^N$ at x.

To compare $\Phi(x)$ w/ $\Phi(y)$ (in a way that's indep of basis choice at each x or y.) we need a connection (comparator):

$$W_{xy} \mapsto \Lambda(x)\, W_{xy}\, \Lambda^{-1}(y)$$

So that $\quad \Phi^\dagger(x) \, W_{xy} \, \Phi(y)$ is inv't.

Demand: $W = W(C_{xy})$

$$\begin{cases} W(\emptyset) \overset{!}{=} \mathbb{1} = W_{xx} \\ W(C_2 \circ C_1) \overset{!}{=} W(C_2) W(C_1) \\ W(-C) = W^{-1}(C) . \end{cases}$$

$$D_\mu \Phi(x) \equiv \lim_{\Delta x \to 0} \frac{W(x, x+\Delta x)\Phi(x+\Delta x) - \Phi(x)}{\Delta x}$$

$$\longmapsto \Lambda(x) D_\mu \Phi(x) .$$

<u>near $\Delta x \to 0$</u>: $\quad W(x, x+\Delta x) \equiv \mathbb{1} - ie \Delta x^\mu A_\mu(x) + \mathcal{O}(\Delta x^2)$

(defn A)

$$D_\mu \Phi \mapsto \Lambda D_\mu \Phi \overset{!}{=} D_\mu^{A^\Lambda}(\Lambda \Phi)$$

$$A \mapsto A^\Lambda = \Lambda A_\mu \Lambda^{-1} - (\partial_\mu \Lambda)\Lambda^{-1} .$$

if $\Lambda = e^{i T^A \lambda^A}$

gives back prev. expression.

$$d^4 l \xrightarrow{\checkmark} d^D l$$

$$\int l^\mu l^\nu g(l^2)\, d^4 l \underset{\uparrow}{\xrightarrow{\checkmark}} \frac{\eta^{\mu\nu}}{D} \int l^2 g(l^2)\, d^D l$$

$$\mathrm{tr}(\mathbb{1}) = 4 \xrightarrow{\text{optional}} 2^{D/2}$$

$$\underline{\gamma_\mu \gamma^\nu \gamma^\mu} = (\epsilon - 2)\gamma^\nu \qquad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$