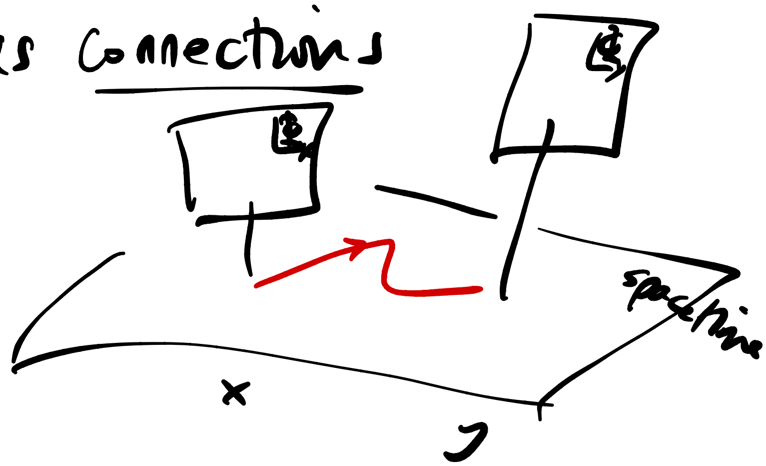
$$\mathrm{tr}\, T^A T^B = \frac{1}{2}\delta^{AB}$$

4.4 Gauge fields as connections



$$\Phi_x \mapsto \Lambda_x \Phi_x$$

$$W(C_{xy}) \mapsto \Lambda_x W(C_{xy}) \Lambda_y^{-1}$$

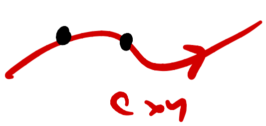
So $\underline{\Phi_x^\dagger W(C_{xy}) \Phi_y}$ is invariant

$$D_\mu \Phi(x) = \lim_{\Delta x \to 0} \frac{W(x, x+\Delta x)\Phi(x+\Delta x) - \Phi(x)}{\Delta x} \mapsto \Lambda_x D_\mu \Phi(x)$$

<u>near $\Delta x^\mu \to 0$</u>: $\begin{cases} \dfrac{\partial W}{\partial x^\mu} = -ie A_\mu(x) W \\ W(p) = \mathbb{1} \end{cases}$ <u>D.E. for W</u>:

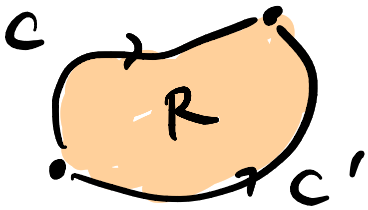
solution is: $W(C_{xy}) = P\, e^{-ie\int_{C_{xy}} A_\mu(\tilde{x}) d\tilde{x}^\mu}$

$\uparrow$ path-ordering $\equiv 1 - e\int A - \frac{e^2}{2} P \iint A\, A + \dots$



To what extent does W depend on the path?

$\boxed{\text{Take } G \text{ abelian}}$



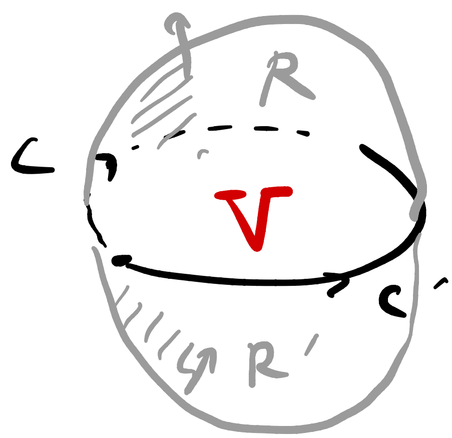
$$W_C = W_{C'}\, e^{ie\int_{C-C'} A}$$

if $C - C' = \partial R$ $\longrightarrow$ $\overset{\text{stokes}}{=} W_{C'}\, \underline{\underline{e^{ie\int_R F}}}$

$F = dA$.

<u>NOT</u>:

$e^{i\oint_{C-C'} A} \equiv$ wilson loop.



<u>The difference is</u>:

$$e^{ie\int_{R-R'} F} \overset{\text{Stokes}}{=} e^{ie\int_V dF}$$

$\partial R = \partial R' = C - C'$

If $F = dA$, A is smooth then $dF = d^2A = 0$.

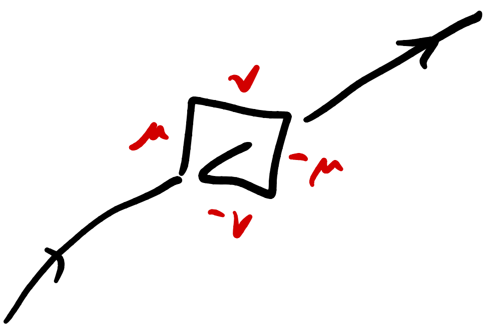
<u>But</u>: $\begin{cases} d{*F} = {*j_e} \\ dF = {*j_m}. \end{cases}$

$$e^{ie\int_V dF} = e^{ie\int_V *j_m} = e^{ie\int_V \rho_m} = e^{ieq} \overset{!}{=} 1$$

$$\iff \boxed{eq \in 2\pi\mathbb{Z}}$$

q = magnetic charge in V.

Dirac quantization condition. $\mathbb{Z}$ = {integers}



$W\Phi \rightsquigarrow$ (no sum on μ,ν.)

$$W\Phi + dx^\mu dx^\nu [D_\mu, D_\nu]\Phi$$

$$= -ie\, dx^\mu dx^\nu F_{\mu\nu}\Phi$$

$$F_{\mu\nu} = \frac{i}{e}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]$$

$F_{\mu\nu}(x) \in \mathfrak{g}$ the Lie algebra of G

T^A are a basis for "

$$F_{\mu\nu}(x) = F^A_{\mu\nu} T^A = (\partial_\mu A^A_\nu - \partial_\nu A^A_\mu - ie f_{ABC} A^B_\mu A^C_\nu) T^A$$

choose $\mathrm{tr}\, T^A T^B = \frac{1}{2}\delta^{AB}$

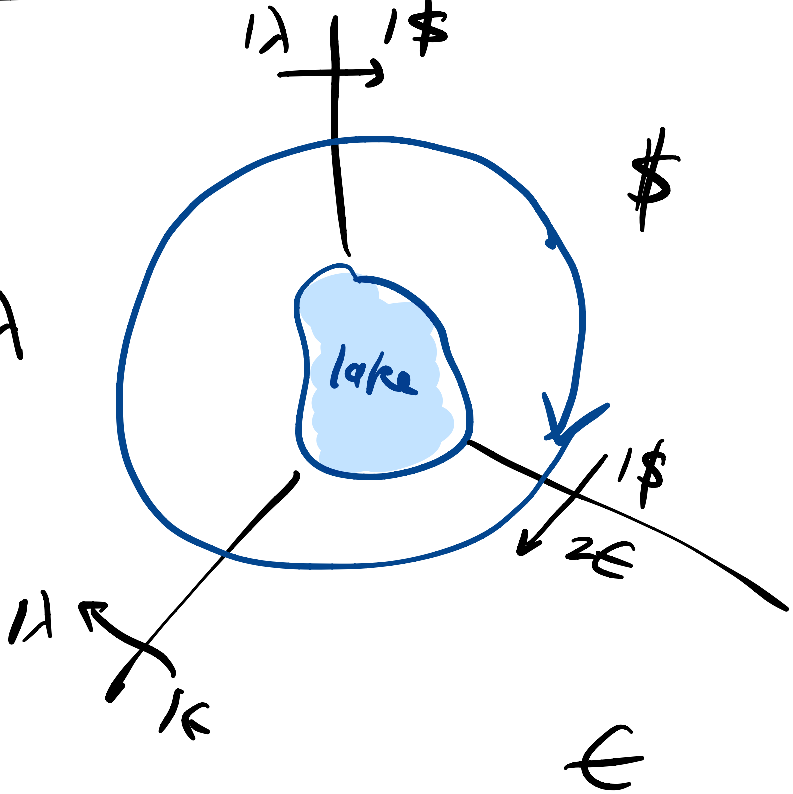
$$F^A = 2\,\mathrm{tr}\, T^A (F).$$

$$[D,D]\Phi_x \longmapsto \Lambda_x [D,D]\Phi_x$$

$$\Rightarrow \quad F(x) \longmapsto \Lambda_x F(x) \Lambda_x^{-1} \quad \text{(adjoint rep)}$$

or $$F^A_{\mu\nu} \mapsto F^A_{\mu\nu} - f^{ABC}\lambda^B F^C_{\mu\nu}.$$

Currency analogy:



- arbitrary value of currency $\longleftrightarrow$ gauge redundancy. λ

- increase of funds around a loop C $\longleftrightarrow e^{i\oint_C A} = e^{i\int F}$

- currency w/ intrinsic value $\longleftrightarrow$ Higgs field

- local obviousness $\longleftrightarrow$ mass for vector.

4.5 Actions for gauge fields

$$S_{YM}[A] = -\frac{1}{2g^2}\int d^Dx \, \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu}$$

- gauge inv't
- Lorentz inv't

$$1 = [D] = [\partial + A] \Rightarrow \underline{[A] = 1}.$$

$$\Longrightarrow [g^2] = 4 - D \qquad \text{(like Maxwell)}$$

$\underline{D=4}$ marginal $\quad$ $\underline{D<4}$: relevant $\quad$ $\underline{D>4}$: irrelevant

<u>Non-abelian</u>: $\mathcal{L}_{YM} \sim (\partial A)^2 + f\underline{A^2\partial A} + ff\underline{A^4}$

self-interaction specified by G.

<u>what else?</u>

<u>D even</u>: $S_\theta = \theta \int \mathrm{tr}\, \underbrace{\frac{F}{2\pi}\wedge\frac{F}{2\pi}\wedge\cdots\wedge\frac{F}{2\pi}}_{D/2 \text{ of these}} = \theta\int d\omega(A)$ is a total <u>derivative</u>

eg $D=4, G=U(1)$: $F\wedge F = d(A\wedge F)$

$$S_\theta = \theta \int_M d\omega \overset{\text{stokes}}{=} \theta \int_{\partial M} \omega$$

$\Longrightarrow$ S_θ doesn't affect EOM
" " pert. theory.

(even in the NA case $\mathrm{tr}\, \frac{F}{2\pi} \wedge \dots = d\omega_{CS}$)

- claim: $\int_M \mathrm{tr}\, \frac{F}{2\pi} \wedge \dots \frac{F}{2\pi} = n \in \mathbb{Z}$. Pf: in discussion of anomalies.
  w/ $\partial M = \phi$ "instanton #"

$$Z_M = \int DA\, e^{i S_0(A) + i S_\theta}$$

$$= \sum_{n \in \mathbb{Z}} \underbrace{\int (DA)_n\, e^{i S_0[A] + i\theta n}}_{Z_n}$$

$$= \sum_{n \in \mathbb{Z}} e^{i\theta n} Z_n .$$

$$\Rightarrow Z_M(\theta) = Z_M(\theta + 2\pi).$$

θ is a periodic variable.

$$\int F \wedge F = \int F_{01} F_{23} \, dt\,dx\,dy\,dz + \dots$$

is odd under CP or T.

In QCD $\theta_{QCD} \neq 0$ produces an EDM for neutrons.

But EDM of neutron $\simeq 0$ in expt.

$\Longrightarrow \theta_{QCD} \leq 10^{-10}$. "strong CP problem".

D odd: $S_{CS}[A] \overset{\text{abelian}}{=} \int A \wedge \underbrace{\frac{F}{2\pi} \wedge \frac{F}{2\pi} \dots}_{\frac{D-1}{2} \text{ of those}}$

In 3d:

$$S_{CS}[A] = \frac{k}{4\pi} \int_M \mathrm{tr}\left(A \wedge F + \frac{2}{3} A \wedge A \wedge A \right)$$

- is not a total derivative $\Rightarrow$ does affect eom.
- Lorentz inv't
- is gauge inv't if $\partial M = \emptyset$.
- breaks P, T.

- k is dimensionless (marginal)

$\mathcal{L}_{YM} \sim D^4$ but $\underbrace{\mathcal{L}_{CS} \sim D^3}_{\text{more relevant}}$.

ERRATUM: g_{YM}^2 is irrelevant in $D<4$

$\mathcal{L}_{CS}$ governs quantum Hall physics.

- is another gauge-inv't way to give the photon a mass.

for $G = U(1)$: $\mathcal{L} = \mathcal{L}_{Max} + \mathcal{L}_{CS} \rightsquigarrow$ massive vector.

In General D: eg: $\int d^D x \, \mathrm{tr}\, F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\mu$

is irrelevant in $D < 6$.

Charged Matter:

$\psi_\alpha(x) \to \Lambda_R(x)_\alpha{}^\beta \psi_\beta(x)$

in some rep R of G.

$$D_\mu \psi = (\underline{\underline{\partial_\mu}} - i \underline{\underline{T_R^A}}\, \underline{\underline{A_\mu^A}}) \psi$$

generators of G in rep R.

$$\mapsto \Lambda_R(x) D_\mu \psi .$$

$$\mathcal{L}_\psi = i \bar{\psi} \gamma^\mu D_\mu \psi - V(\bar{\psi}\psi)$$

OR

$$\mathcal{L}_\Phi = D_\mu \bar{\Phi} D^\mu \Phi - V(\bar{\Phi}\Phi)$$

gauge-inv't

Lorentz inv't.

<u>choices</u>: $\left\{\begin{array}{l} G \\ \text{Reps of fermions} \\ \text{Reps of Scalars} \\ \text{self-couplings of ferms \& scalars.} \end{array}\right.$

4.6 Fermion path integrals

$$\underline{\{\psi(x), \bar{\psi}(y)\}} = i\hbar\,\delta^d(x-y) \xrightarrow{\hbar\to 0} 0$$

canonical (anti) commutation rel'ns

Grassmann variables $\{\theta_i \quad i = 1\ldots n\}$

$$\longrightarrow \begin{cases} \theta_i\theta_j = -\theta_j\theta_i \\ \theta_i^2 = 0 \end{cases}$$

$\boxed{\text{Realization: coord. differentials } dx^i \wedge dx^j.}$

multiply & add w/ $\mathbb{C}$ coeffs $\rightarrow$ Grassman algebra $=$ even $\oplus$ odd

For $n=1$: $\quad g(\theta) = a + b\theta$

For $n=2$: $\quad \underline{g(\theta_1,\theta_2)} = a + b\theta_1 + c\theta_2 + d\,\underline{\theta_1\theta_2}$

$[\text{even}, \text{even/odd}] = 0 \qquad \{\text{odd}, \text{odd}\} = 0.$

$b = \frac{\partial g}{\partial\theta_1} \qquad c = \frac{\partial g}{\partial\theta_2}$

Taylor's thm for Grassman vars $\qquad d = \frac{\partial^2 g}{\partial\theta_1\partial\theta_2}.$

Integration = differentiation for grassmann vars

$$\boxed{\text{Integration Table}: \quad \int \psi \, d\psi = 1 \qquad \int 1 \, d\psi = 0}$$

$$1 = \int \bar{\psi}\psi \, d\psi \, d\bar{\psi} = -\int \bar{\psi}\psi \, d\bar{\psi} \, d\psi$$

$$\int d\psi_1 \ldots d\psi_n \, \chi(\psi\ldots) = \partial_{\psi_1} \cdots \partial_{\psi_n} \chi \, .$$

$$\int_{-\infty}^{\infty} dx \, f(x) = \int_{-\infty}^{\infty} dx \, f(x+a) \quad \text{if } \partial_x a = 0$$

$$\int (A + B\theta) \, d\theta = \int d\theta \, (A + B(\theta + \alpha)) \, d\theta \, .$$

if $\partial_\theta \alpha = 0$.

$$\boxed{\int \underbrace{e^{-a\bar{\psi}\psi}}_{1 - a\bar{\psi}\psi} d\bar{\psi} \, d\psi = +a \, .}$$

Many: $\bar{\psi}\cdot A\cdot\psi =$

$$(\bar{\psi}_1 \ \ldots \ \bar{\psi}_M)\begin{pmatrix} A_{11} & A_{12} & \\ & \ddots & \\ & & \ddots \end{pmatrix}\begin{pmatrix}\psi_1\\ \vdots \\ \psi_M\end{pmatrix}$$

$$\int e^{-\bar{\psi}\cdot A\cdot\psi}\prod_i^M d\bar{\psi}_i \prod_i^M d\psi_i$$

$$=\int\left(1-\bar{\psi}A\psi+\frac{1}{2}\bar{\psi}A\psi\,\bar{\psi}A\psi+\ldots\right)\pi\, d\bar{\psi}\,\pi\, d\psi$$

$$=\frac{1}{n!}\sum_{\text{perms},\sigma}(-1)^{\sigma}A_{1\sigma_1}A_{1\sigma_2}\cdots A_{m\sigma_m}$$

$$=\det(A)=e^{+\operatorname{tr}\log A}$$

fermion loop (-1).

Compare:

$$\int e^{-\phi^* A\phi}\,d\phi^* d\phi=\frac{\#}{\det A}=e^{-\operatorname{tr}\log A}$$

one:

$$\langle \bar\psi \psi \rangle \equiv \frac{\int \bar\psi \psi \; e^{-a\bar\psi\psi} d\bar\psi d\psi}{\int e^{-a\bar\psi\psi} d\bar\psi d\psi} = -\frac{1}{a}$$

$$= -\langle \psi \bar\psi \rangle$$

many: $S = \sum_i a_i \bar\psi_i \psi_i$ (diagonalize A)

$$\langle \bar\psi_i \psi_j \rangle = \frac{\delta_{ij}}{a_i} \equiv \langle \bar{i} j \rangle$$

or: $\langle \bar\psi_i \psi_j \rangle = (A^{-1})_{ij}$

Wick's Thm: $\langle \bar\psi_i \bar\psi_j \psi_k \psi_l \rangle = \langle \bar{i} l \rangle \langle \bar{j} k \rangle$

$$- \langle \bar{i} k \rangle \langle \bar{j} l \rangle.$$

complete square

$$\int e^{-\bar\psi \cdot A \cdot \psi + \bar\eta_i \psi_i + \bar\psi_i \eta_i} \prod d\bar\psi_i \prod d\psi_i =$$

$$= e^{\bar\eta A^{-1} \eta} \int \exp - (\bar\psi - \bar\eta A^{-1}) A (\psi - A^{-1}\eta)$$

$$= e^{\bar\eta A^{-1} \eta} \det A.$$

$\psi_i \rightsquigarrow \psi(x)$

$f(\psi_i) \rightsquigarrow f[\psi]$

$\boxed{\text{choose } A \text{ to discretize } i\not\partial - m}$

$$Z[\bar\eta,\eta] = \int [D\bar\Psi D\Psi]\, e^{i\int d^Dx\,\left[\bar\Psi(i\not\partial - m)\Psi + \bar\eta\Psi + \bar\Psi\eta\right]}$$

$$\equiv \prod_i d\bar\Psi_i \prod_i d\Psi_i$$

$$= \det(i\not\partial - m)\,\exp i\int_x\int_y \bar\eta_y (i\not\partial - m)^{-1}\eta_x$$

$$\bar\Psi_1\bar\Psi_2\bar\Psi_3\Psi_1\Psi_2\Psi_3 = -\bar\Psi_1\Psi_1\bar\Psi_2\Psi_2\bar\Psi_3\Psi_3$$

$$\bar\Psi_1\bar\Psi_2\Psi_1\Psi_1 = -\bar\Psi_1\Psi_1\bar\Psi_2\Psi_2$$