5.1 cont'd: Feynman Rules for Non-Abelian Gauge Theory

$$Z_{YM} = \int [DA^a Dc^a D\bar{c}^a]\, e^{\,i\int \left(\mathcal{L}_{YM} - \frac{(\partial\cdot A)^2}{2\xi} + \bar{c}(-\partial\cdot D)c\right)}$$

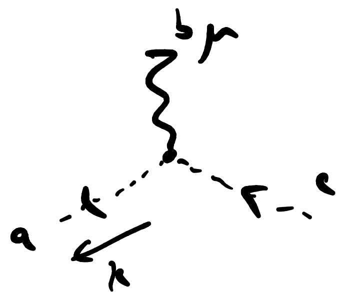
$$\mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b A^c$$

$$a,b,c = 1 \ldots \dim G$$

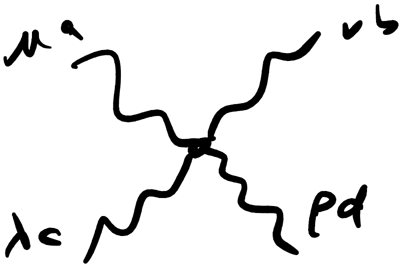
$$\langle A^a_\mu(x) A^b_\nu(y) \rangle = \delta^{ab}\,(\text{photon propagator})_{\mu\nu}$$

$$= \frac{i}{k^2}$$



$$= -g f^{abc} k^\mu$$

$$= g f^{abc}\left(\eta_{\mu\nu}(k_1-k_2)_\lambda + \eta_{\nu\lambda}(k_2-k_3)_\mu + \eta_{\lambda\mu}(k_3-k_1)_\nu\right)$$



$$= -ig^2 \Big(f^{abe} f^{cde} (\eta_{\mu\lambda}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\lambda})$$

$$+ \quad (b,\nu) \longleftrightarrow (d,\rho)$$

$$+ \quad (b,\nu) \longleftrightarrow (c,\lambda) \Big)$$

$$Z_{QCD} = \int [DA^a Dc^a D\bar{c}^a Dq^i D\bar{q}^i]\, e^{iS_{YM} + i\int \mathcal{L}_{quarks}}$$

$i = 1 .. \dim R$

R is a rep of G

$$\mathcal{L}_{quarks} = \bar{q}(i\not{D} - m) q$$

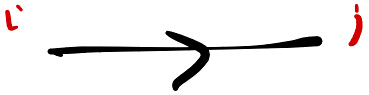
$$= \bar{q}_i \left[\gamma^\mu (i\partial_\mu \delta_{ij} + g A^a_\mu t^a_{ij}) - m\,\delta_{ij} \right] q_j$$

eg: $i = 1..N$ fundamental of $G = SU(N)$.

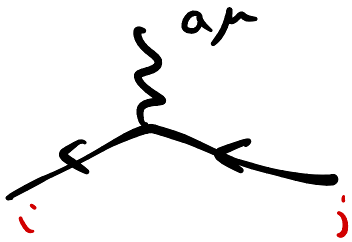
for $N=3$ t^a_{ij} Gell-Mann Matrices.

omitted: flavor indices & coupling to E&M

$$\bar{q}^\alpha_i Q_\alpha e A_\mu q^\alpha_i \qquad Q_\alpha = \begin{cases} 2/3 & u \\ -1/3 & d \end{cases}$$



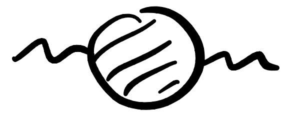
$= \delta^{ij}$ (e^- propagator)



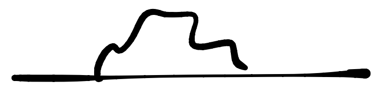
$= i g \gamma^\mu t^a_{ij}$

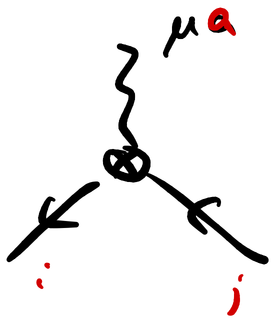
Counterterms:

$= -i(k^2\eta^{\mu\nu} - k^\mu k^\nu)\delta^{ab}\delta_3 \leftarrow$

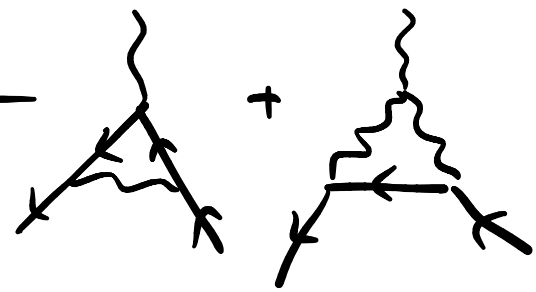


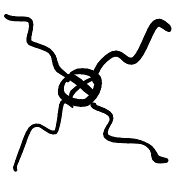
$= i \not{k} \delta_2 \leftarrow$



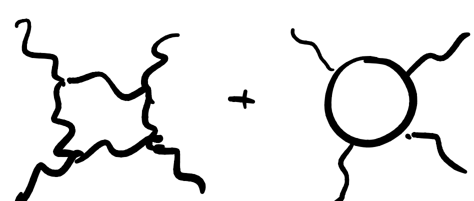


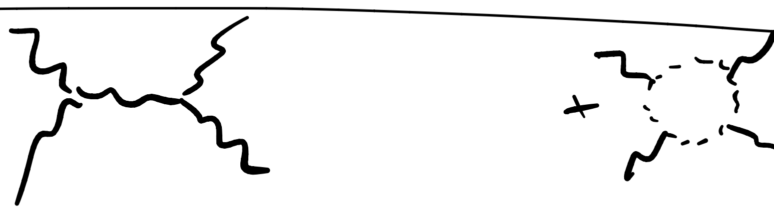
$= i g t^a_{ij} \gamma^\mu \delta_1 \leftarrow$





$= g^2 \delta_4 (\ldots\ldots) \leftarrow$





$+ \ldots ?$

5.2 QCD Beta f'n

dim reg, $\overline{MS}$: $\beta(g_R) = \mu \partial_\mu g_R$

choose δ's to subtract the $\frac{1}{\epsilon}$ poles. $\epsilon = 4-D$.

free Lagrangian (quadratic)

$$\mathcal{L} = -\frac{1}{4} Z_3 (\partial A)^2 + Z_2 \bar{q}(i\not{\partial} - Z_R m_R) q - Z_c \bar{c}^a \Box c^a$$

$$- \mu^{\epsilon/2} g_R Z_{A^3} f^{abc} (\partial_\mu A_\nu^a) A^{\mu b} A^{\nu c}$$

$$- \frac{\mu^\epsilon}{4} g_R^2 Z_{A^4} (f^{abc} A_\mu^b A_\nu^c)(f^{ade} A^{\mu d} A^{\nu e})$$

$$- \mu^{\epsilon/2} g_R Z_1 A_\mu^a \bar{q} \gamma^\mu t^a q + \mu^{\epsilon/2} g_R Z_{3c} f^{abc} \partial_\mu \bar{c}^a A_\mu^b c^c$$

$Z_x = 1 - \delta_x$ four counterterms for g: Z_1, Z_{A^3}, Z_{A^4}, Z_{3c}.

Bare fields $A_\mu^0 = \sqrt{Z_3} A_\mu$, $q^0 = \sqrt{Z_2}\, q$, $c^0 = \sqrt{Z_c}\, c$

have quadratic terms w/o Z's:

$$(\partial A^0)^2 + \bar{q}^0 i \not{\partial} q^0 + \bar{c}^0 \Box c^0$$

$$\mathcal{L}_{qqg} = \underbrace{\mu^{\frac{4-D}{2}} g_R Z_1 Z_3^{-1/2} Z_2^{-1}}_{\equiv\, g_0 \,=\, \mu^{\frac{4-D}{2}} \frac{Z_1}{Z_2\sqrt{Z_3}} g_R} A_\mu^{0a} \bar{q}^0 \gamma^\mu t^a q^0$$

BARE Coupling

Powers of μ: $[g_R]=0$ in D dims.

$$\tilde{A} = gA \longrightarrow -\frac{1}{4g^2}\mathrm{tr}F^2$$

$$S = -\underbrace{\int d^Dx}_{-D} \frac{\mu^{D-4}}{4g_R^2} \underbrace{\mathrm{tr}\,F^2}_{+4} \qquad [F]=2$$

$$\boxed{g \longrightarrow \mu^{\epsilon/2} g.}$$

"Callan-Symanzik eqn": bare coupling doesn't know about μ.

$$0 = \mu\frac{\partial}{\partial\mu} g_0 = \mu\frac{\partial}{\partial\mu}\left(\mu^{\epsilon/2} g_R \frac{Z_1}{Z_2\sqrt{Z_3}}\right) =$$

$$0 \overset{!}{=} g_0 \left(\frac{\epsilon}{2} + \frac{1}{g_R} \underbrace{\mu \frac{\partial}{\partial\mu} g_R}_{\beta(g_R)} + \mu\partial_\mu \left(\delta_1 - \frac{\delta_3}{2} - \delta_2\right)\right) + \mathcal{O}(g^3)$$

$$Z_x^\alpha = (1-\delta_x)^\alpha$$
$$= 1 - \alpha\,\delta_x + \mathcal{O}(\delta^2)$$

solve for β!

$$\delta \sim \mathcal{O}(g^2)$$

$$\beta(g_R) = -\frac{\epsilon}{2} g_R - g_R\,\mu\partial_\mu \left(\delta_1 - \frac{\delta_3}{2} - \delta_2\right) + \mathcal{O}(g^4)$$

CLAIM:

$$\delta_x = \frac{g_R^2}{\epsilon} \#_x + \mathcal{O}(g^3)$$

$$\mu\partial_\mu \delta_x = \mu \frac{\partial g_R}{\partial \mu} \frac{\partial \delta_x}{\partial g_R} = \beta(g_R) \frac{\partial \delta_x}{\partial g_R}$$

$$= -\frac{\epsilon}{2} g_R - g_R \underbrace{\beta(g_R)}_{-\frac{\epsilon}{2} g_R + \mathcal{O}(g_R^2)} \frac{\partial}{\partial g_R}\left(\delta_1 - \frac{\delta_3}{2} - \delta_2\right) + \mathcal{O}(g^4)$$

$$= -\frac{\epsilon}{2} g_R + \frac{\epsilon}{2} g_R^2\, \partial_{g_R}\left(\delta_1 - \frac{\delta_3}{2} - \delta_2\right) + \mathcal{O}(g_R^4)$$

$$\mathcal{L}_{A^3} = \underbrace{\mu^{\epsilon/2} g_R Z_{A^3} Z_3^{-3/2}}_{g_0} f^{abc}\, \partial_\mu A^a_\nu A^{b\mu} A^{c\nu}$$

$\longrightarrow$ same answer.

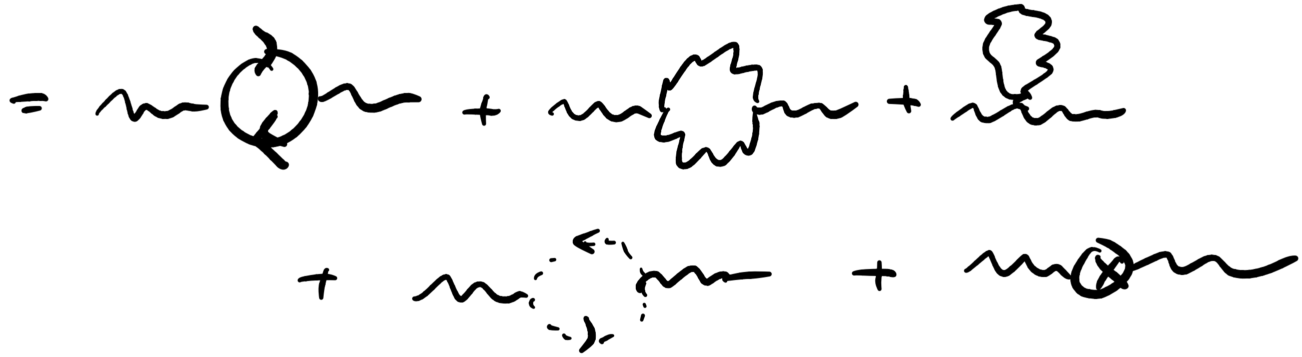
$\longrightarrow$

$$\beta(g_R) = -\frac{\epsilon}{2} g_R + \frac{\epsilon}{2} g_R^2\, \partial_{g_R}\left(\delta_{A^3} - \frac{3}{2}\delta_3\right)$$

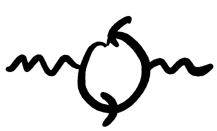
Need to know: $\delta_{1,2,3}$ through $O(g^2)$.

Gluon vac. polarization: Ward id $\Rightarrow$

$$i\Pi^{\mu\nu}_{ab}(q) = -i\,\underline{\underline{\Pi_{ab}(q^2)}}\,(q^2\eta^{\mu\nu} - q^\mu q^\nu)$$



$$= \mathcal{M}_q + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_{ghost}$$
$$-i(k^2\eta^{\mu\nu} - k^\mu k^\nu)\,\delta^{ab}\,\underline{\delta_3}.$$

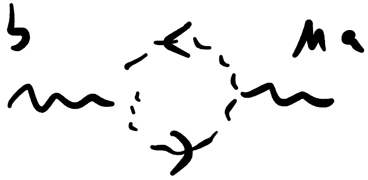
$i\mathcal{M}_q^{\mu\nu ab} =$  $= \underline{\underline{\mathrm{tr}_R\, t^a t^b}}\; i\mathcal{M}^{\mu\nu}_{QED}(e\to g)$

$R=F$: $\mathrm{tr}_F\, t^a t^b = T_F\,\delta^{ab} = \frac{1}{2}\delta^{ab}$.

Quark masses are relevant ops, don't affect UV behavior.

WLOG set $m_q = 0$.

$$i\mathcal{M}_q^{\mu\nu ab} \overset{D\to 4}{=} N_f T_F (q^2\eta^{\mu\nu} - q^\mu q^\nu)\,\delta^{ab}\,\frac{g^2}{16\pi^2}\left(-\frac{8}{3}\frac{1}{\epsilon} - \frac{20}{9} - \frac{4}{3}\ln\frac{\mu^2}{-q^2} + O(\epsilon)\right)$$

$iM^{\mu\nu ab}_{ghost} =$  $= (-1)(-g)^2 \int d^Dk \cdot$

$$\longrightarrow \quad \frac{i}{(k-q)^2}\,\frac{i}{k^2}\, \underline{\underline{f^{cad}}}\, k^{\mu}\, \underline{\underline{f^{dbc}}}\,(k-q)^{\nu}$$

Recall: $J^2 = \vec{J}\cdot\vec{J} = j(j+1)\mathbb{1}$ on spin j rep of $SU(2)$

quadratic casimir:

$$(T^2)_{ij} = (T^a T^a)_{ij} \quad \text{satisfies}$$

$$[T^b, T^2] = 0 \quad \forall\, b.$$

Schur's lemma

$$\Longrightarrow \quad (T^a T^a)_{ij} = C_2(r)\,\mathbb{1}_{d(r)\times d(r)}$$

$$T^a_{adj}\, T^a_{adj} = -(f^a)_{bc}(f^a)_{cd}$$

$$= f^{abc} f^{acd} = C_2(G)\,\delta^{bd}$$

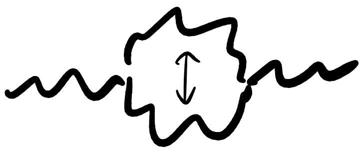
$$\mathrm{tr}\, T^a_r T^b_r = C(r)\,\delta^{ab}$$

for $SU(N)$

$$\longrightarrow \quad \boxed{\dim(r)\, C_2(r) = \dim G\, C(r).} \qquad \underline{C_2(G) = N.}$$

$$i\mathcal{M}^{\mu\nu ab}_{ghost} = g^2 \frac{\bar{\mu}^{4-D}}{(4\pi)^{D/2}} \delta^{ab} C_2(G) \int_0^1 dx \left(\frac{1}{\Delta}\right)^{2-D/2} \left[\eta^{\mu\nu}\left(\frac{1}{2}\Gamma\left(2-\frac{D}{2}\right)\Delta\right) + q^\mu q^\nu \left(x(1-x)\,\Gamma(2-D/2)\right)\right]$$

$$\Delta = x(x-1)q^2 .$$

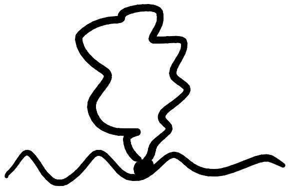
$$i\mathcal{M}_3^{\mu\nu ab} =$$  $$= \frac{g^2}{2}\bar{\mu}^{4-D} \int d^D k \times \frac{-i}{k^2}\frac{-i}{(k-q)^2} f^{acd} f^{bcd} N^{\mu\nu}$$

$$= -\frac{g^2}{2}\frac{\bar{\mu}^{4-D}}{(4\pi)^{D/2}} \delta^{ab} C_2(G) \int dx \left(\frac{1}{\Delta}\right)^{2-D/2} \left(\eta^{\mu\nu}A + q^\mu q^\nu B + \eta^{\mu\nu} q^2 C\right)$$

$$\Delta = x(x-1)q^2$$

quadratic divergence.

$\longrightarrow$ pole at $D=2$.



$$= i\mathcal{M}_4^{\mu\nu ab} = \frac{ig^2}{2}\bar{\mu}^{4-D}\int d^Dk\,\frac{-i}{k^2}\,\eta^{\rho\lambda}\,\delta^{(cd)}\Big[$$

$$f^{abe}\,\cancel{f^{[cde]}} \to 0\;\left(\delta^{\mu}_{b}\delta^{\nu}_{\rho} - \delta^{\mu}_{\rho}\delta^{\nu}_{\lambda}\right)$$

$$+\,(b\nu) \longleftrightarrow (d\rho)$$

$$+\,(b\nu) \longleftrightarrow (c\lambda)\Big] \times 2$$

$$= -g^2\,\delta^{ab}\,C_2(G)\,(D-1)\,\bar{\mu}^{4-D}\int \frac{d^4k}{k^2}\,\frac{(q-k)^2}{(q-k)^2}$$

$$= -g^2\,\delta^{ab}\,C_2(G)\,\eta^{\mu\nu}\,(D-1)\,\bar{\mu}^{4-D}\int dx\left(\frac{1}{\Delta}\right)^{2-D/2}\Big[-\frac{D}{2}\Gamma\!\left(1-\frac{D}{2}\right)\Delta - (1-x)^2q^2\,\Gamma\!\left(2-\frac{D}{2}\right)\Big]$$

$$\mathcal{M}^{\mu\nu ab}_{glue}(q) = \left(\mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_{ghost}\right)^{\mu\nu ab}$$

$$= \delta^{ab}\,C_2(G)\,g^2\,\frac{\bar{\mu}^{4-D}}{(4\pi)^{D/2}}\int dx\left(\frac{1}{\Delta}\right)^{2-D/2}\Big[\,\Gamma^{\mu\nu}\,\Big]$$

$$\Gamma^{\mu\nu} = \eta^{\mu\nu}\, \Gamma\!\left(1-\tfrac{D}{2}\right) \Delta \underbrace{\left(-\tfrac{1}{2} + \tfrac{3(D-1)}{2} - \tfrac{D(D-1)}{2}\right)}_{= -\frac{1}{2}(D-2)^2}$$

pole at $D=2$

$$+ q^\mu q^\nu\, \Gamma\!\left(2-\tfrac{D}{2}\right) a + \eta^{\mu\nu} q^2\, \Gamma\!\left(2-\tfrac{D}{2}\right) b$$

$$\Gamma\!\left(1-\tfrac{D}{2}\right)(D-2) = -2\,\Gamma\!\left(2-\tfrac{D}{2}\right) \leftarrow \quad \Gamma(1+x) = x\,\Gamma(x)$$

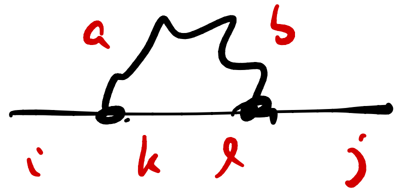
$$\mathcal{M}^{\mu\nu ab}_{glue}(q) \overset{D=4-\epsilon}{=} C_2(G)\,\delta^{ab}\left(\eta^{\mu\nu} - q^\mu q^\nu\right) \frac{g^2}{16\pi^2} \times \left[\boxed{\frac{10}{3}\frac{1}{\epsilon}} + \frac{31}{9} + \frac{5}{3}\ln\frac{\mu^2}{-q^2} + O(\epsilon)\right]$$

$\overline{MS}$

$$\Rightarrow \delta_3 = \frac{1}{\epsilon}\frac{g^2}{16\pi^2}\left[\underbrace{\frac{10}{3}C_2(G)}_{\text{glue}} - \underbrace{\frac{8}{3}N_f T_f}_{\text{quarks}}\right]$$

Quark self energy : $\longrightarrow \delta_2, \delta_m$

$(m_q = 0)$

$i\Sigma_2^{ij}(\not{p}) =$ 

$$= \int d^Dk \; t^a_{ij}\gamma^\mu \frac{i\not{k}\,\delta^{kl}}{k^2+i\epsilon}\, t^b_{lj}\gamma_\mu \frac{-i\delta^{ab}}{(k-p)^2+i\epsilon}$$

$$t^a_{ik}\, t^b_{lj}\, \delta^{ab}\delta^{kl} = \sum_a (t^a t^a)_{ij} = C_2(R)\delta_{ij}.$$

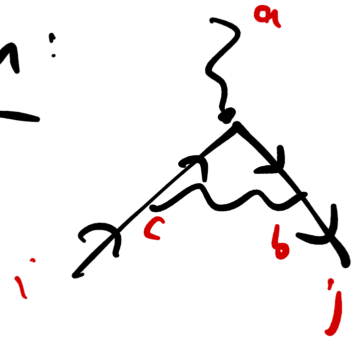
$$\text{tr (RHS)} \xRightarrow[\text{of } SU(N)]{\text{for } R=F} C_2(F) = \frac{N^2-1}{2N}.$$

$$\Sigma_2^{ij}(\not{p}) = \frac{g^2}{8\pi^2}\delta^{ij}C_2(F)\frac{1}{\epsilon}\not{p} + \text{finite}$$

$$\Sigma^{ij} = \dots + \delta_2 \not{p}\, \delta^{ij}$$

$$\xRightarrow{\overline{MS}} \delta_2 = \frac{1}{\epsilon}\frac{g^2}{16\pi^2}\left(-2C_2(F)\right).$$

Vertex Correction:



$$= i g \Gamma^{\mu}_{QED}(e \to m) \, (t^c t^a t^b)_{ij} \times \delta^{bc}$$

$$t^c t^a t^b \delta^{bc} = t^b t^a t^b = \underbrace{t^b t^b}_{C_2(F)\mathbb{1}} t^a + t^b \underbrace{[t^a, t^b]}_{i f^{abc} t^c}$$

$$i f^{abc} t^b t^c = i f^{abc} \frac{1}{2} [t^b, t^c] = -\frac{1}{2} f^{abc} f^{bcd} t^d$$

$$= -\frac{1}{2} C_2(G) t^a$$

$$= + i g \left(C_2(F) - \frac{1}{2} C_2(G) \right) t^a_{ij} \gamma^{\mu} \frac{g^2}{16\pi^2} \left(\frac{2}{\epsilon} + \ln \frac{\mu^2}{p^2} + \text{finite} \right)$$

$$= i g f^{abc} (t^c t^b)_{ij} \Gamma^{\mu}_{new}(p^2)$$

$$-ig\Gamma^{\mu}_{new}(p^2) = (ig)^2 g\bar{\mu}^{4-D}\int d^Dk\, \gamma_\rho \frac{i}{k\!\!\!/}\gamma_\nu \frac{i}{(q+k)^2}\frac{i}{(q'-k)^2}$$

$$\times \underbrace{N^{\mu\nu\rho}(k,q,q')}_{= 3\text{ glue vertex}}$$

we want $\frac{1}{\epsilon}$ indep. of external momenta!

$$-\Gamma^{\mu}_{new}(p^2\to 0) = g^2\bar{\mu}^{4-D}\int d^Dk \cdot \frac{\gamma_\rho k\!\!\!/ \gamma_\nu}{k^6}\left(\eta^{\mu\nu}k^\rho - 2\eta^{\nu\rho}k^\mu + \eta^{\rho\mu}k^\nu\right)$$

$$\gamma_\rho\gamma^\nu\gamma^\rho = (2-D)\gamma^\nu$$

$$\left\{\int k^\mu k^\nu f(k^2) = \int \frac{k^2}{D} f(k^2).\right.$$

$$= g^2\bar{\mu}^{4-D}\gamma^\mu\left(4-\frac{4}{D}\right)\int\frac{d^Dk}{k^4}$$

$$= \frac{g^2 i\gamma^\mu}{16\pi^2}\left(\frac{6}{\epsilon} + 3\log\frac{\mu^2}{-p^2} + \text{finite}\right)$$

$$f^{abc}(t^c t^b) = \frac{1}{2} i f^{abc} f^{cbd} t^d = -\frac{i}{2}C_2(G)t^a$$

$$0 = \frac{1}{\epsilon} ig\, t^a_{ij} \gamma^\mu \Big(2\big(C_2(F) - \tfrac{1}{2} C_2(G) \big) + 3 C_2(G) \Big) \times \frac{g^2}{16\pi^2} + \delta_1 \epsilon \Big)$$

$$\Rightarrow \quad \delta_1 = \frac{1}{\epsilon} \frac{g^2}{16\pi^2} \left(-2 C_2(F) - 2 C_2(G) \right).$$

$$\beta(g) = -\frac{\epsilon}{2} g + \frac{\epsilon}{2} g^2 \partial_g \left(\delta_1 - \frac{\delta_3}{2} - \delta_2 \right) + O(g^4)$$

$$\overset{D\to 4}{=} \frac{g^3}{16\pi^2} \Big(-2 \cancel{C_2(F)} - 2 C_2(G) - \frac{1}{2} \Big(\frac{10}{3} C_2(G) - \frac{8}{3} N_f T_f \Big) - \big(-2 \cancel{C_2(F)} \big) \Big)$$

$$= -\frac{g^3}{16\pi^2} \left(\frac{11}{3} C_2(G) - \frac{4}{3} N_f T_f \right) + O(g^4)$$

$G = SU(N)$
w/ $\square$ quarks

$$= -\frac{g^3}{16\pi^2} \left(\frac{11}{3} N - \frac{2}{3} N_f \right).$$

$\begin{pmatrix} T_f = 1/2 \\ C_2(G) = N. \end{pmatrix}$

if $N_f < 6N \overset{N=3}{=} 18$, $\underline{\underline{\beta < 0.}}$

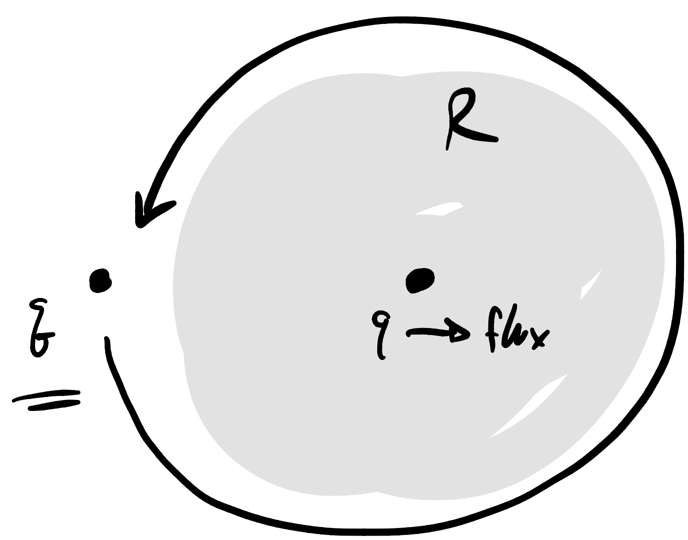
$$\frac{\delta}{\delta A_\mu(x)}\left[\left(-\frac{1}{4}\right)\int d^3x \; \underline{\underline{F_{\rho\sigma}F^{\rho\sigma}}} + \int A_\nu J^\nu\right]$$

$$= \partial_\nu F^{\mu\nu} + J^\mu$$

$$\underline{\underline{A \wedge F}} = A_\mu F_{\nu\rho} \; \underline{\underline{dx^\mu \wedge dx^\nu \wedge dx^\rho}}$$

$$\propto \epsilon^{\mu\nu\rho} \; \underline{\underline{d^3x}}$$

$dx\,dy\,dt$



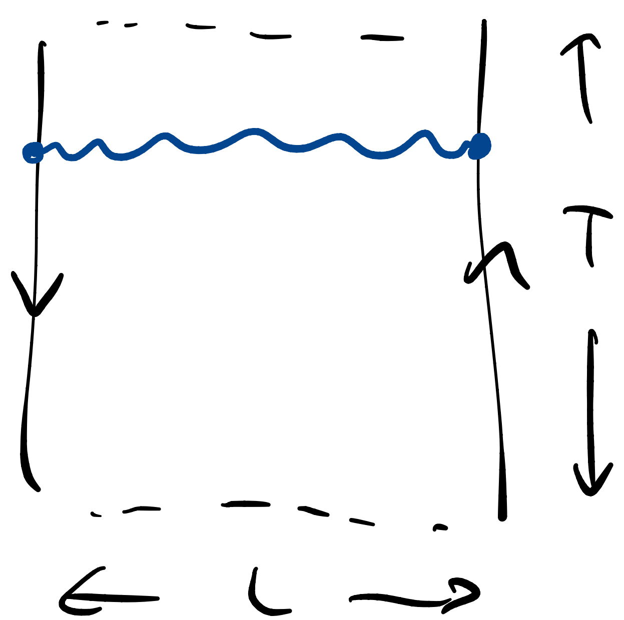
$$e^{iq\oint_{\underline{C}} a} = e^{iq\int_R \text{flux}}$$

$$\underline{\underline{\omega_p \sim \sqrt{p^2+m^2}}}\Big|_{p=0} = m.$$

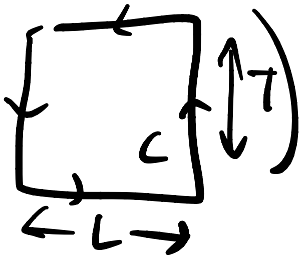
$$A_\mu = \underline{\underline{\varepsilon_\mu(k)}}\, e^{ikx}$$

$$0 = \mathrm{eom}\left(\underline{\underline{\varepsilon_\mu}},\ \underline{\underline{k_\mu}},\ \epsilon_{\mu\nu\rho}\right)$$

$$\langle A_\mu(x)\, A_\nu(0)\rangle = \int d^D k\, \frac{e^{ikx}}{k^2-m^2}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}\right)$$



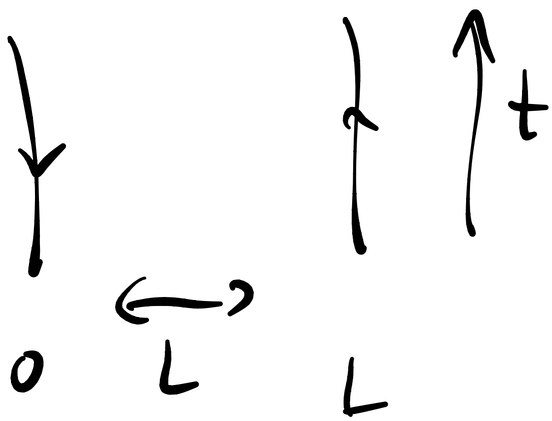
Abelian:

$\langle W($  $)\rangle$

$$= \cancel{\frac{1}{Z}} \int [DA]\, e^{-S[A]}\, \underbrace{e^{i\oint_C A}}\, e^{i\int A_\mu j^\mu}$$

want: the bit which is $e^{-T(\#)}$

$$j^\mu_{(x)} = \int dt\, \delta^4(x - X^\mu(t)) = \delta^{\mu 0}\left(\delta^3(x) - \delta^3(x-L)\right)$$



$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\int (DA)\, e^{-\int \left((\partial A)^2 - Aj \right)}$$

$$\overset{\text{gaussian}}{=} \underline{\underline{\#}}\, e^{\int_{xy} j_x D_{xy} j_y}$$