University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 213 Fall 2023
Assignment 7 – Solutions

Due 11:00am Tuesday, February 28, 2023

1. **Brainwarmer.** Check that the Holevo quantity $\chi(p_a, \rho_a) = S(\sum_a p_a \rho_a) - \sum_a p_a S(\rho_a)$ can be written as a relative entropy
$$\chi(p_a, \rho_a) = D(\rho_{AB} || \rho_A \otimes \rho_B)$$
with $\rho_{AB} \equiv \sum_a p_a \rho_a \otimes |a\rangle\langle a|$ where $B = \text{span}\{|a\rangle\}$ and the $|a\rangle$ are orthonormal.

2. **Shannon entropy is concave.** Consider a collection of probability distributions π^α on a random variable X, so $\sum_x \pi^\alpha_x = 1, \pi^\alpha_x \geq 0, \forall x$. Then a convex combination of these $\pi_{\text{av}} \equiv \sum_\alpha p_\alpha \pi^\alpha$ is also a probability distribution on X. Show that the entropy of the average distribution is larger than the average of the entropies:
$$H(\pi_{\text{av}}) \geq \sum_\alpha p_\alpha H(\pi^\alpha).$$

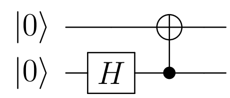
The most direct method is to note that $-x \log x$ is a concave function, so that Shannon entropy is concave.

A slightly less direct method is to notice that
$$H(\pi_{\text{av}}) = -\sum_x \sum_\alpha p_\alpha \pi^\alpha_x \log\left(p_\beta \pi^\beta_x\right) \tag{1}$$
and by concavity of the logarithm $-\log\left(p_\beta \pi^\beta_x\right) \geq -\sum_\beta p_\beta \log \pi^\beta_x \geq -\sum_\beta \log \pi^\beta_x \geq -\log \pi^\alpha_x$ for any α. Therefore
$$H(\pi_{\text{av}}) = -\sum_x \sum_\alpha p_\alpha \pi^\alpha_x \log\left(p_\beta \pi^\beta_x\right) \tag{2}$$
$$\geq -\sum_x \sum_\alpha p_\alpha \pi^\alpha_x \log \pi^\alpha_x = \sum_\alpha p_\alpha H(\pi^\alpha). \tag{3}$$

Another method is to relate the difference to a relative entropy, as suggested by the previous problem:
$$H(\pi_{\text{av}}) = -\sum_x \sum_\alpha p_\alpha \pi^\alpha_x \log\left(p_\beta \pi^\beta_x\right) \tag{4}$$
$$= -\sum_x \sum_\alpha p_\alpha \pi^\alpha_x \log\left(\pi^\alpha_x\right) - \sum_x \sum_\alpha p_\alpha \pi^\alpha_x \log\left(\frac{p_\beta \pi^\beta_x}{\pi^\alpha_x}\right) \tag{5}$$
$$= \sum_\alpha p_\alpha H(\pi_\alpha) - D(\pi_{\text{av}} || \pi^\alpha) \geq \sum_\alpha p_\alpha H(\pi^\alpha) \tag{6}$$

using $D(\cdot||\cdot) \geq 0$.

3. **Making a Bell pair from a product state.** Find the output of the following quantum circuit (time goes to the right here and in the following):



Here $H = \frac{1}{\sqrt{2}}(X + Z)$ is a Hadamard gate, and the two-qbit gate is the CX gate as in lecture.

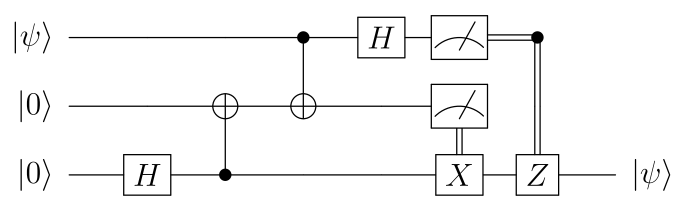
Use 1 and 2 to label the top and bottom qbits respectively. The control X distinguishes control and target bits:

$$\mathsf{CX}_{ct} = |0\rangle\langle 0|_c \mathbb{1}_t + |1\rangle\langle 1|_c \mathbf{X}_t.$$

Then the circuit is doing

$$\mathsf{CX}_{21} \cdot \mathbf{H}_2 |00\rangle = \mathsf{CX}_{21}\left(\frac{|00\rangle + |01\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right).$$

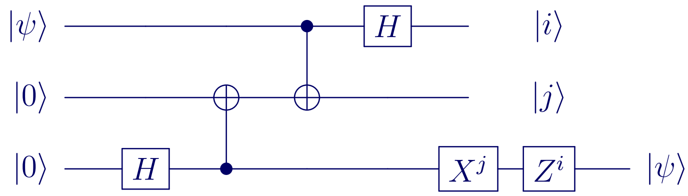
4. **Quantum Teleportation.** Convince yourself that it is possible to transmit an unknown state of a qbit by sending two classical bits to someone with whom you share a Bell pair, using the following circuit:



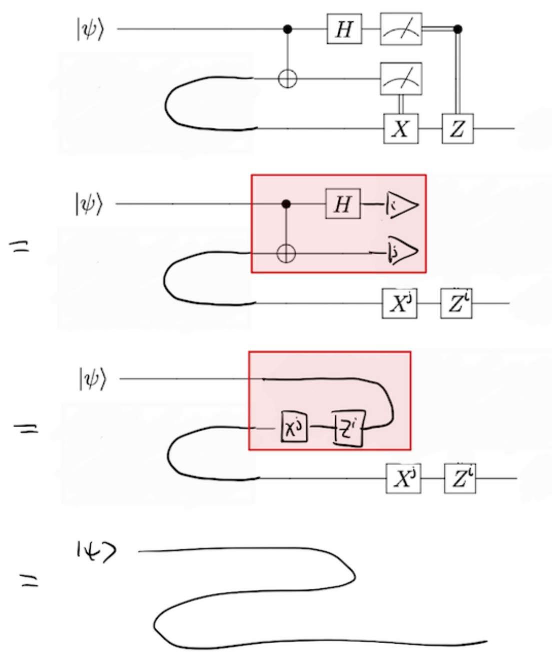
Time goes from left to right here; you should recognize the first two operations from the previous problem. Imagine that the register on the bottom line is separated in space from the top two after this point. The measurement boxes indicate measurements of Z; the double lines indicate that the outcomes of these measurements $s = 1, 0$ (this is the sending of the two classical bits) determine whether or not (respectively) to act with the indicated gate.

A solution to this problem and the next can be found on pages 1-58 – 1-60 here. But here is a better explanation.

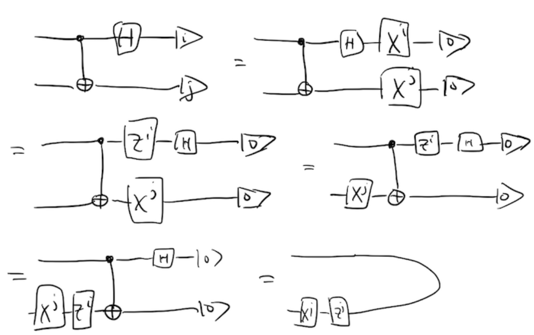
A nice way to think about the measurement step (that I learned from Ting-Chun (David) Lin and Yu-Hsueh Chen) is the following. For each possible outcome of the measurement, just feed that state into the circuit:



We can also use the previous problem to simplify the circuit. The first two gates just prepare a maximally-entangled state between B and C:



The step highlighted in the red box is explained here:

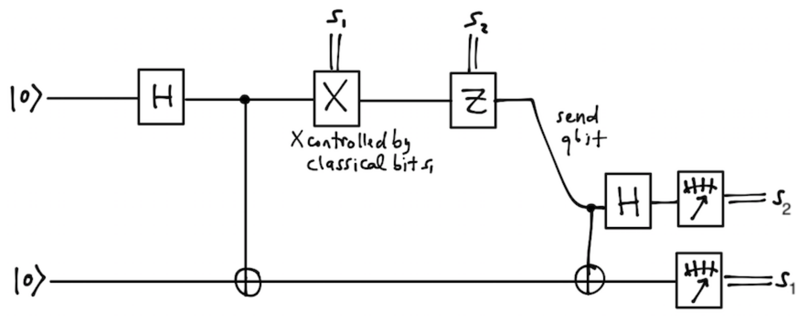


Here I used the algebra between X, Z and CX, which is quite useful:

$$\mathsf{CX}_{12}Z_1 = Z_2\mathsf{CX}_{12}, X_2\mathsf{CX}_{12} = \mathsf{CX}_{12}Z_2.$$

5. **Quantum Dense Coding.** Find a circuit which does the reverse of the previous: by sending an unknown qbit to someone with whom you share a Bell pair, transmit two classical bits. (Hint: basically just reverse everything in the previous problem.)

   Using the same notation as on the previous problem, a circuit diagram for the solution (credit to Ziyi Zhao for drawing it this way) looks like:



   After the first two gates, we've produced a Bell pair between qbits 1 and 2. Acting with $Z_1^{s_2} X_1^{s_1}$ gives

$$\sqrt{2}\,|\Psi_1\rangle = X_1^{s_1} Z_1^{s_2} \left(|00\rangle + |11\rangle\right) = (-1)^{s_1 s_2}\,|s_1 0\rangle + (-1)^{\bar{s}_1 s_2}\,|\bar{s}_1 1\rangle\,.$$

   Acting with CX_{12} gives

$$\sqrt{2}\,|\Psi_2\rangle = \mathsf{CX}_{12}\,|\Psi_1\rangle = (-1)^{s_1 s_2}\,|s_1 s_1\rangle + (-1)^{\bar{s}_1 s_2}\,|\bar{s}_1 s_1\rangle = \left((-1)^{s_1 s_2}\,|s_1\rangle + (-1)^{\bar{s}_1 s_2}\,|\bar{s}_1\rangle\right) \otimes |s_1\rangle\,.$$

   Acting with H_1 gives

$$\begin{aligned}
\mathsf{H}_1\,|\Psi_2\rangle &= \frac{1}{2}\left((-1)^{s_1 s_2}\left(|0\rangle + (-1)^{s_1}|1\rangle\right) + (-1)^{\bar{s}_1 s_2}\left(|0\rangle + (-1)^{\bar{s}_1}|1\rangle\right)\right) \otimes |s_1\rangle && (7)\\
&= \frac{1}{2}\left[\left((-1)^{s_1 s_2} + (-1)^{\bar{s}_1 s_2}\right)|0\rangle + \left((-1)^{s_1 \bar{s}_2} + (-1)^{\bar{s}_1 \bar{s}_2}\right)|1\rangle\right] \otimes |s_1\rangle && (8)\\
&= \frac{1}{2}\left[\left(1 + (-1)^{s_2}\right)|0\rangle + \left(1 + (-1)^{\bar{s}_2}\right)|1\rangle\right] \otimes |s_1\rangle && (9)\\
&= |s_2\rangle \otimes |s_1\rangle && (10)
\end{aligned}$$

   A simple check is that if $s_1 = s_2 = 0$, the circuit is $HCXCXH = HH = 1$.

6. **Teleportation for qdits.** [optional]

   Show that it is possible to teleport a state $|\xi\rangle_A \in \mathcal{H}_A, |A| \equiv d$ from A to B using the maximally-entangled state

$$|\Phi\rangle_{AB} \equiv \frac{1}{\sqrt{d}} \sum_{n=1}^{d} |nn\rangle_{AB}\,.$$

Hint: Consider the clock and shift operators

$$\mathbf{Z} \equiv \sum_{n=1}^{d} |n\rangle \langle n| \omega^n, \ \omega \equiv e^{\frac{2\pi \mathbf{i}}{d}}, \quad \mathbf{X} \equiv \sum_{n=1}^{d} |n+1\rangle \langle n|$$

where the argument of the ket is to be understood mod d. Show that these generalize some of the properties of the Pauli $\mathbf{X}$ and $\mathbf{Z}$ in that they are unitary and that they satisfy the (discrete) Heisenberg algebra

$$\mathbf{XZ} = a\mathbf{ZX}$$

for some c-number a which you should determine.

7. **Conditional entropy in terms of relative entropy.**

   (a) Show that the conditional entropy can be written as

   $$S(A|C) = -D(\rho_{AC}||\mathbb{1}_A \otimes \rho_C). \tag{11}$$

   (b) Does the relation (11) imply that the conditional entropy can never be positive? Find a proof or a counterexample.

   The correct answer is 'no': the positivity of the relative entropy $D(\rho||\sigma) \geq 0$ depends on ρ, σ being normalized density matrices. The object $1_A \otimes \rho_C$ is not a normalized density matrix on AC.

   The conditional entropy can be positive, as it is for any classical distribution on AC

   $$\rho_{AC} = \sum_{ac} p_{ac} |ac\rangle\langle ac|,$$

   for example. Another counterexample is $\rho_{AC} = \rho_A \otimes \rho_C$, where $S(A|C) = S(AC) - S(C) = S(A) \geq 0$.

8. **It's a trap.** Is the mutual information convex?

$$I_{\sum_a p_a \rho_a}(A:B) \overset{?}{\leq} \sum_a p_a I_{\rho_a}(A:B)$$

It's a relative entropy, and the relative entropy is jointly convex in its arguments, right? Find a proof or a counterexample.

A counterexample is a state of the form

$$\rho = \sum_c p_c \rho^c = \sum_c p_c |cc\rangle\langle cc|$$

where the states $|cc\rangle$ are orthonormal product states on AB. Then $S_{\rho^c}(A) = S_{\rho^c}(B) = S_{\rho^c}(AB) = 0$, so $\sum_c p_c I_{\rho^c}(A:B) = 0$. On the other hand $S_A(\sum_c p_c |cc\rangle\langle cc| = S(\sum_c p_c |c\rangle\langle c|) = H(p)$, and $S_B = S_{AB} = H(p)$ as well, so $I_\rho(A:B) = H(p)$ is nonzero.

The joint convexity of the relative entropy says that for $\rho = \sum_c p_c \rho^c_{AB}$

$$
\begin{aligned}
I_\rho(A:B) &= D(\rho_{AB}||\rho_A \otimes \rho_B) && (12)\\
&= D(\sum_c p_c \rho^c_{AB}|| \sum_c p_c \rho^c_A \otimes \sum_{c'} p_{c'} \rho^{c'}_B) && (13)\\
&= D(\sum_{cc'} p_c p_{c'} \rho^c_{AB}|| \sum_{cc'} p_c p_{c'} \rho^c_A \otimes \rho^{c'}_B) && (14)\\
&\overset{\text{joint convexity}}{\leq} \sum_{cc'} p_c p_{c'} D(\rho^c_{AB}||\rho^c_A \otimes \rho^{c'}_B) && (15)\\
&- \sum_{cc'} p_c p_{c'} \left(\mathrm{tr}\rho^c_{AB} \log \rho^c_{AB} - \mathrm{tr}\rho^c_{AB} \log \rho^c_A \otimes \rho^{c'}_B \right) && (16)\\
&= \sum_c p_c \mathrm{tr}\rho^c_{AB} \log \rho^c_{AB} - \sum_c p_c \mathrm{tr}\rho^c_A \rho^c_A - \sum_{cc'} p_c p_{c'} \mathrm{tr}\rho^c_B \log \rho^{c'}_B && (17)\\
&= \sum_c p_c \left(\mathrm{tr}\rho^c_{AB} \log \rho^c_{AB} - \mathrm{tr}\rho^c_A \log \rho^c_A - \mathrm{tr}\rho^c_B \log \rho^c_B + \mathrm{tr}\rho^c_B \log \rho^c_B \right) - \sum_{cc'} p_c p_{c'} \mathrm{tr}\rho^c_B \log \rho^{c'}_B && (18)\\
&= \sum_c p_c I_{\rho^c}(A:B) + \sum_{cc'} p_c p_{c'} \left(\mathrm{tr}\rho^c_B \log \rho^c_B - \mathrm{tr}\rho^c_B \log \rho^{c'}_B \right) && (19)\\
&= \sum_c p_c I_{\rho^c}(A:B) + \sum_{cc'} p_c p_{c'} D(\rho^c_B||\rho^{c'}_B). && (20)
\end{aligned}
$$

The extra term in (20) is strictly positive. Moreover, the relative entropy $D(\rho||\sigma)$ is unbounded (it is infinity when ρ has support where σ does not).

This problem is dedicated to Tarun Grover.