University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 213 Winter 2023
Assignment 10

Due 11:00am Tuesday March 21, 2023

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1. **Simple stabilizer codes.**

   (a) Consider the Hamiltonian on two qbits

   $$-H = X_1 X_2 + Z_1 Z_2 .$$

   Show that the terms commute and that the groundstate is

   $$\frac{|00\rangle + |11\rangle}{\sqrt{2}} .$$

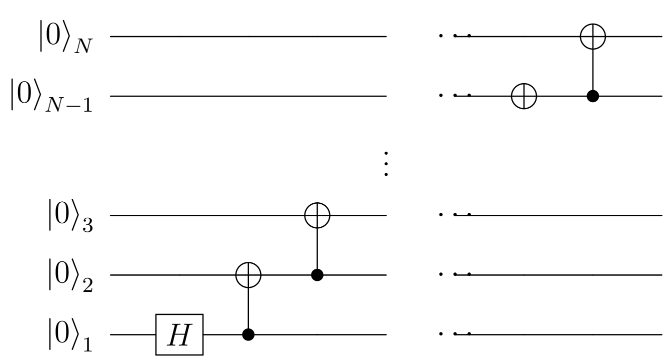
   (b) Consider the (non-local) Hamiltonian on N qbits

   $$H_{\rm GHZ} = -X_1 \cdots X_N - \sum_{i=1}^{N-1} Z_i Z_{i+1} . \tag{1}$$

   Show that all the terms commute. Show that the groundstate is (the GHZ state)

   $$\frac{|00...0\rangle + |11...1\rangle}{\sqrt{2}} .$$

   (c) Show that the following circuit U produces the GHZ state from the product state $|0\rangle^{\otimes N}$.



   (d) What state does U produce from $|1\rangle_1 \otimes |0\rangle^{\otimes N-1}$?

(e) Find the result of feeding the Hamiltonian $-\sum_i Z_i$ (whose groundstate is the product state $|0\rangle^{\otimes N}$) through the circuit, *i.e.* what is

$$U\left(-\sum_i Z_i\right)U^\dagger \quad ?$$

Hint: use the rules for the action of CX by conjugation given in lecture.

(f) **The five-qubit perfect code.** [optional but fun] Consider the Hamiltonian on five qubits

$$H \equiv -\left(ZXXZ\mathbb{1} + \text{cyclic permutations}\right). \tag{2}$$

Show that $X_L = XXXXX$ and $Z_L = ZZZZZ$ are suitable logical operators. Construct the two groundstates. One way to do this is to start with $|00000\rangle$ and act with the projector onto the code subspace. The projector onto the code subspace is

$$\Pi = \prod_{B\in S}\left(\frac{1+B}{2}\right) = \frac{1}{2^5}\prod_{s\in\mathsf{F}_2^5} B_1^{s_1}\cdots B_5^{s_5} \tag{3}$$

where S is a set of generators the stabilizer group (terms in H), and by F_2^5 I just mean strings of 5 bits. Then

$$|0_L\rangle = \frac{1}{2^5}\sum_{s_1\cdots s_5=0,1} B_1^{s_1}\cdots B_5^{s_5}\,|00000\rangle\,. \tag{4}$$

Bonus problem: show that the groundstate wavefunction $A_{z_1\cdots z_5} \equiv \langle z_1\cdots z_5|0_L\rangle$ is a *perfect tensor*. This means that it is maximally entangled with respect to all possible bipartitions.

Super-bonus problem: Study the physics of the model with $H = -\sum_i Z_i X_{i+1} X_{i+2} Z_{i+3}$ in the thermodynamic limit.

2. **Algebraic condition for stabilizer code.** [optional] We can represent a Hamiltonian on q qbits, where each term is a product of Xs and Zs, by a $2q\times T$ matrix σ, where T is the number of terms in the hamiltonian. (This is the transpose of the object I wrote in lecture.) Each column represents a term in the Hamiltonian. The top q rows indicate where the Zs are and the bottom q rows indicate where the Xs are. Think of it as a map from the set of stabilizers (terms in H) to the set of Pauli operators.

For example, the matrix for the example in problem 1a is

$$\sigma_{1a} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \quad .$$

Convince yourself that the condition for all the terms to commute is that

$$\sigma^t \lambda \sigma = 0 \text{ mod } 2$$

where

$$\lambda \equiv \begin{pmatrix} 0 & 1\!\!1_{q\times q} \\ 1\!\!1_{q\times q} & 0 \end{pmatrix} .$$

Check that this is the case for the examples above.

For a beautiful elaboration of this machinery that incorporates translation invariance, see Haah's thesis.

3. **Bekenstein bound.** [optional]

   In this problem, $\hbar = c = k_B = 1$.

   (a) A black hole has a temperature $T_{BH} = \frac{1}{8\pi G_N M}$ and (in Einstein gravity) an entropy $S_{BH} = \frac{A}{4G_N}$, where $A = 4\pi R^2$ is the area of the event horizon, and $R = 2G_N M$ is the Schwarzchild radius. Check that this is consistent with the first law of thermodynamics $dE = TdS$, where $E = M$.

   (b) The generalized second law then says that $S_{\text{total}} = S_{BH} + S_{\text{stuff}}$ is non-decreasing. Suppose we have an object of linear size R (say it fits in a sphere of radius R) whose energy E and entropy S satisfy $S \overset{?}{>} 2\pi ER$. Then we can cram some extra stuff in there until the object undergoes gravitational collapse and forms a black hole. Convince yourself that this would violate the generalized second law. Thus we arrive at the *Bekenstein bound*, $S \leq 2\pi ER$. Notice that G_N has dropped out of this relation. Indeed a version of it follows simply from positivity of the relative entropy (see below).

   (c) [optional bonus part which requires some general relativity] To understand why a black hole has a temperature, notice that near the horizon at $r = 2G_N M$, the Schwarzchild metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right), \;\; f(r) = 1 - \frac{2G_N M}{r}$$

looks like

$$ds^2_{\text{Rindler}} = -\kappa^2\rho^2 dt^2 + d\rho^2 + r_s^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{5}$$

where $\rho = 2\sqrt{r_s(r-r_s)}$ for a constant κ, and $r_s \equiv 2G_N M$. Determine κ.

Show that regularity of this geometry in euclidean time $\tau \equiv \mathbf{i}t$ requires periodic euclidean time $\tau \simeq \tau + \beta(\kappa)$. Find $\beta(\kappa)$ and interpret it as an inverse temperature.

Moreover, show that in the coordinates $T = \kappa\rho\sinh\eta, Z = \rho\cosh\eta$ (with $\eta \equiv \kappa t$), the near-horizon metric (5) is

$$ds^2_{\text{Rindler}} = -dT^2 + dZ^2 + R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

namely $\mathbb{R}^{1,1} \times S^2$. However, only the region $Z > 0$ describes the region outside the horizon. This means that the system outside the horizon must be described by a density matrix which traces out the region $Z < 0$.

Now recall from lecture the Bisognano-Wichmann theorem: in the ground state of a relativistic quantum field theory, the entanglement Hamiltonian for a half-space cut is the boost generator

$$K = 2\pi\int_{x>0} dx x T_{00}.$$

That is, the reduced density matrix $\rho_0 = e^{-K}/\text{tr}e^{-K}$ is a thermal state with Hamiltonian K. Moreover, the Rindler rapidity η is proportional to the asymptotic Minkowski time coordinate t. Check that the temperature obtained this way agrees with the euclidean periodicity argument.

(d) Show that a version of the Bekenstein bound can be obtained from positivity of the relative entropy. More precisely, consider some region of space, and write the reduced density matrix of the vacuum state as $\rho_0 = \frac{e^{-K}}{\text{tr}e^{-K}}$. Show that $0 \leq D(\rho||\rho_0)$ can be written as

$$S(\rho) - S(\rho_0) \leq \text{tr}\rho K - \text{tr}\rho_0 K.$$

Interpret the left hand side as the entropy above the vacuum, and the RHS as $(E - E_0)R$ where E_0 is the vacuum energy.

4. **LOCC versus entanglement creation.** [Optional. This problem was suggested to me by Tarun Grover.] A fact that has received a lot of recent attention in the literature and in the press is that one can prepare the toric code groundstate by starting with a product state, measuring the toric code stabilizers (the terms in the toric code hamiltonian), and correcting any errors that occur. That

is, if one finds that some of the stabilizers gives -1 instead of $+1$, this means that there are anyon excitations; one can then group these anyons in pairs, move them toward each other using local unitaries, and annihilate them.

The toric code groundstate is highly entangled. For example, if we divide the plane into two halves A and B, there will be nontrivial long-range entanglement between A and B.

On the other hand, a desideratum for *any* measure of entanglement between A and B is that the entanglement does not increase under LOCC operations.

Identify which step or steps above cannot be done with LOCC between A and B. (Suppose for definiteness that each of A and B is a fixed collection of links of the lattice on which the toric code is defined.)