University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 213 Winter 2023
Assignment 11 ("Why?")

Due never

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1. **Majorization questions.**

   (a) Show that if a doubly stochastic map is reversible (invertible and the inverse is also doubly stochastic) then it is a permutation.

   (b) Show that the set of doubly stochastic maps is convex (that is: a convex combination $\sum_a p_a D_a$ of doubly stochastic maps is doubly stochastic). What are the extreme points of this set? (This is the easier direction of the Birkhoff theorem.)

   (c) Show that a pure state and uniform state satisfy $(1, 0, 0 \cdots) \succ p \succ (1/L, 1/L \cdots)$ for any p on an L-item space.

   (d) A useful visualization of majorization relations is called the 'Lorenz curve': this is just a plot of the cumulative probability $P_p(K) = \sum_{k=1}^{K} p_k^{\downarrow}$ as a function of K. What does $p \succ q$ mean for the Lorenz curves of p and q? Draw the Lorenz curves for the uniform distribution and for a pure state.

   (e) Show that the set of probability vectors majorized by a fixed vector x is convex. That is: if $x \succ y$ and $x \succ z$ then $x \succ ty + (1-t)z, t \in [0, 1]$. Hints: (1) the analogous relation is true if we replace x, y, z with real numbers and $\succ$ with $\geq$. (2) Show that $P_{p^{\downarrow}}(K) \geq P_{\pi p^{\downarrow}}(K)$ (where $\pi p^{\downarrow}$ indicates any other ordering of the distribution).

   (f) For the case of a 3-item sample space we can draw some useful pictures of the whole space of distributions. The space of probability distributions on three elements is the triangle $x_1 + x_2 + x_3 = 1, x_i \geq 0$, which can be drawn in the plane. We can simplify the picture further by ordering the elements $x_1 \geq x_2 \geq x_3$, since majorization does not care about the order. Pick some distribution x with $x_1 \neq x_2 \neq x_3$ and draw the set of distributions which x majorizes, the set of distributions majorized by x, and the set of distributions with which x does not participate in a majorization relation ('not comparable to x').

2. **Checking the operational interpretation of trace distance.**

(a) **Warmup.** Show that for two pure states $|1\rangle, |2\rangle$, their trace distance T and their fidelity F satisfy

$$F^2 + T^2 = 1.$$

(b) In lecture we proved a result relating the probability of success at distinguishing two states by a single measurement to their trace distance. Might it be possible to evade this theorem by considering POVMs which are not projective measurements?

Consider two non-orthogonal pure states $|1\rangle, |2\rangle$. with overlap $\delta = |\langle 1|2\rangle|^2$ and consider the POVM made of :

$$E_1 = \chi |1\rangle \langle 1|, E_2 = \alpha |2\rangle \langle 2|, E_3 = 1 - E_1 - E_2.$$

For which values of χ, α is this a POVM?

Find the probability of success of the strategy: if outcome is 1 guess 1, if outcome is 2 guess 2, if outcome is 3 do a little dance then guess randomly. Show that the bound we proved is not violated.

(c) Nevertheless, POVMs (which are not projective measurements) are indeed useful for state discrimination. Find a POVM with the property that distinguishes between two non-orthogonal pure states $|1, 2\rangle$ in such a way that for one outcome we are *certain* that the state is $|1\rangle$ and for another we are *certain* that the state is $|2\rangle$. (There is a third outcome where we learn nothing from the measurement.)

3. **Entanglement negativity for pure states.** Show that when $\rho_{AB} = |\psi\rangle\langle\psi|$ is pure, the logarithmic negativity

$$E_N(\rho_{AB}) \equiv \log \| \rho^{T_A} \|_1$$

is the Renyi entropy of index 1/2, $S_{1/2}(\rho_A)$, with $S_\alpha(\rho) \equiv \frac{1}{1-\alpha} \log \mathrm{tr} \rho^\alpha$. Use the Schmidt decomposition.

4. **Additivity of squashed entanglement.** [from Preskill]

(a) Use the chain rule for mutual information and the non-negativity of the conditional mutual information to show that

$$I(AA' : BB'|C) \geq I(A : B|C) + I(A' : B'|AC). \tag{1}$$

Conclude that the squashed entanglement is *superadditive*, *i.e.*

$$E_{sq}(AA' : BB') \geq E_{sq}(A : B) + E_{sq}(A' : B').$$

(b) Show that for the special case of product states of the form $\rho_{ABA'B'} = \rho_{AB} \otimes \rho_{A'B'}$, the inequality (1) is saturated:

$$E_{sq}(AA' : BB') \overset{\text{product states}}{=} E_{sq}(A : B) + E_{sq}(A' : B').$$

5. **Literature quest.** [optional] In lecture I mentioned some sufficient conditions (something like additivity, convexity) for an entanglement monotone $E_X(\rho)$ to satisfy

$$E_D(\rho) \leq E_X(\rho) \leq E_F(\rho)$$

where E_D, E_F are the entanglement of distillation and formation respectively. Find the right conditions and a proof that they are sufficient in the literature (or, more ambitiously, find them yourself).