The Reeh-Schlieder Theorem: Unraveling Intuition

Anthony Vizcaíno Aportela*
University of California San Diego

(Phys 213)

(Dated: March 24, 2023)

This paper presents a discussion of the Reeh-Schlieder theorem and its implications in the context of quantum field theory (QFT) by challenging the intuitive understanding of the physical universe. The discussion begins by questioning the assumption that local operators cannot generate certain states and arguing that any state can be approximated by applying clever combinations of local operators on the vacuum state. This is demonstrated through a series of mathematical derivations and thought experiments, highlighting the highly entangled nature of the vacuum state. The paper also addresses the apparent paradox arising from the theorem's implications and clarifies that it does not provide a way to physically alter the universe in extraordinary ways. By examining the Reeh-Schlieder theorem, the paper emphasizes the importance of understanding the properties of local operators in QFT and their role in describing the universe.

I. INTRODUCTION

In an effort to build intrigue, I introduce the Reeh-Schlieder theorem in a sort of backwards way. Instead of starting with the statement of the theorem, I lull the reader into a false sense of security by giving a description of what is seemingly an intuitive fact about physics. After this I will try to prove that it's at least wrong in the context of QFT and the Reeh-Schlieder theorem will fall out from that discussion.

II. THIS SECTION IS A LIE

Through learned experience in classical physics, or just in the study of systems of differential equations, one wouldn't be faulted for believing that the evolution of the universe, past present and future, can be described by it's equations of motion and enough initial conditions. And by the converse, if we don't have enough initial conditions, then we don't have a universe. In our universe "enough initial contitions" would be a full spanning slice of space-time where the state is known. We might call it the 4D hyper-plane at $t = 0$, or some other translated, rotated, and/or boosted plane.

So let's say we have a quantum field theory (QFT) in some 1+d Minkowski space-time and wanted to know what the probability was that the vacuum state transforms to some other state using only local operators. Since these operators are only local, then we naively assume that there must exist some states that can only be reached if we acted on the vacuum state using explicitly non-local operators. Then the probability of reaching this state would be zero, *i.e.* it would be orthogonal to any state reached locally.

More precisely, let there be a Minkowski space $\mathcal{M}$ with coordinates $\mathbf{x} := (t, \vec{x})$, and a couple of states $|\Omega\rangle$ and $|\chi\rangle$ in Hilbert space $\mathcal{H}$. Let there be a local region of space-time $\mathcal{O}$ with a set of local operators $s := \{\phi(\mathbf{x}) : \mathbf{x} \in \mathcal{O}\}$. If $|\Omega\rangle$ is the vacuum state and $|\chi\rangle$ is some nontrivial state supposedly orthogonal to anything transformed from the vacuum using only local operators, then the correlation function between the two states,

$$\langle\chi| \phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\ldots\phi(\mathbf{x}_n) |\Omega\rangle = 0,$$

for any subset in s. Let's summarize succinctly;

If the universe works the way we think it does, then there must exist some nontrivial $|\chi\rangle$ *such that* $\langle\chi| \phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\ldots\phi(\mathbf{x}_n) |\Omega\rangle = 0$.

III. A MILD DISCOMBOBULATION IN INTUITION

For fun, let's see what happens if we translate the last operator in the correlation by some arbitrary time,

$$e^{\mathbf{i}Ht}\phi(\mathbf{x}_n)e^{-\mathbf{i}Ht}.$$

Since $H|\Omega\rangle = 0$, we can define a correlation function

$$g(t) = \langle\chi| \phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\ldots e^{\mathbf{i}Ht}\phi(\mathbf{x}_n) |\Omega\rangle .$$

Note that this implies the following:

1. $g(0) = 0$
2. $g(t)$ is holomorphic in the upper half t-plain
3. $g(t)$ is continuous on the real axis

from which we can induce that $g(t)$ is identically zero **everywhere**. In other words, translating $\phi((x)_n)$ to a different operator, local or otherwise, doesn't stop the correlation function from being zero.

We can keep going and ascribe the same procedure to $\phi(\mathbf{x}_{n-1})$ as well and define a new correlation function,

$$g'(t) = \langle\chi| \phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\ldots e^{\mathbf{i}Ht}\phi(\mathbf{x}_{n-1})\phi(\mathbf{x}_n) |\Omega\rangle .$$

* aaportel@ucsd.edu

By the same thought process, we can show that $g'(t)$ is also identically zero. If we do it to every single local operator, we get that no matter what, the correlation function vanishes. This argument works for any arbitrary space-time vector because we can arbitrarily boost the time vector. To recap, we shifted every local operator by some arbitrary vector, and still got that the correlation is zero, implying that $|\chi\rangle$ has to be the trivial zero vector. In other words, the only state that is orthogonal to the set of states transformed from the vacuum state by local operators is the zero vector.

The implication here is that all states, even those generated by non-local operators, can be generated by a sufficiently clever set of local operators. Conceptually, this means that there is enough information stored locally in the vacuum state to generate the rest of the universe. Instead of an entire hyperplane of space, we only need an arbitrarily small sub-region of space to describe the universe.

This is the manifestation of the Reeh-Schlieder theorem, that **any** state of the universe can be approximated by operating sufficiently clever combinations of local operators on the vacuum state.

IV. AN APPARENT PARADOX

Consider a state of the universe at some time-slice that looks like the vacuum state near some local region A. Now let's say there's a galaxy in some space-like separated region B. We can assign an operator SW instantiating the entire *Star Wars* literary canon in region B. If the universe is in a state which caters to my fantasies, then the expectation of SW is close to 1, and 0 otherwise.

$$\langle\Omega'|SW\,|\Omega'\rangle \sim 1$$
$$\langle\Omega|SW\,|\Omega\rangle \sim 0$$

The Reeh-Schlider theorem states that there is some operator X in region A which when acted on the vacuum, produces the *Star Wars* literary cannon in region B.

$$X\,|\Omega\rangle = |\Omega'\rangle$$

Then we can write

$$\langle\Omega|\,X^\dagger(SW)X\,|\Omega\rangle = 1.$$

Since SW and X are space-like separated, we can commute them to get

$$\langle\Omega|\,(SW)X^\dagger X\,|\Omega\rangle\,.$$

But wait a minute, wouldn't $X^\dagger X$ cancel out, giving us $1 = 0$? Nope. The theorem make's no statement about the unitarity of our local operators. And since we can only physically realize unitary operators, this prevents us from generating any of these arbitrary fantasy fulfilling operators in a lab.

Although this theorem doesn't let us alter the universe in any exceedingly new and unusual way, there is still something interesting to be learned here. *The vacuum state is by definition highly entangled.*

ACKNOWLEDGMENTS

This paper closely follows Edward Witten's *Notes on Some Entanglement Properties of Quantum Field Theory*[1] Further discussion and application to entanglement entropy can be found there.

[1] E. Witten, Aps medal for exceptional achievement in research: Invited article on entanglement properties of quantum field theory, Reviews of Modern Physics **90**, 045003 (2018).