Active Error Correction of Zero Temperature Topological Quantum Memory

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In this paper we will discuss the error correcting properties of the toric code. We discuss the critical probability where error correction is guaranteed and how zero temperature improves error correction. Information given in PHYS 213 will be taken as given, but a brief refresher is provided. This is by no means a comprehensive guide to the subject, but presents a few choice pieces of information on the subject.

INTRODUCTION

Again, introductory knowledge of the toric code is taken as given, but some is reiterated here for review and convenience. The toric code was designed by A. Kitaev for the purpose of quantum computation [1]. For a 2D graph with qubits located on each lattice link, we have two check operators, the plaquette operator and site operator, respectively[2][3][4]

$$Z_P = \prod_{l \in P} Z_l, \;\; X_s = \prod_{l \in s} X_l, \tag{1}$$

where the plaquette measures the 4 links surrounding a square and site the 4 links surrounding a vertex, so each check will measure four qubits. Plaquette operators check for bit flips while site operators check for phase flips. These operators are local, meaning that they only affect neighboring qubits.

It is also important for this paper to define chains. A 1-chain is any mapping which assigns an element of Z_2=$\{0,1\}$ to each lattice link (where a qubit is located), and sometimes also refers to all links which are mapped to the value 1. 0-chains map 0, or 1 to a lattice site and 2-chains map the same to each plaquette. [2].

ERROR CORRECTION

The toric code has an impressive way of finding and correcting errors. The syndrome is comprised of the values of all check operators on the plane. If there are no errors, the values from the check operators will be +1, however in the presence of errors they will be -1. If errors occur along a series of links, they are called an error chain and the sites on the boundaries are referred to as defects. There are many possible error chains that can create this boundary, such as those in figure 1, but fortunately error chains with the same boundary generate the same syndrome and can be identically corrected by acting upon them with the Z operator. This can be thought of as laying down a "recovery chain" between the two defects and acting on it, which can recover the code's quantum state.

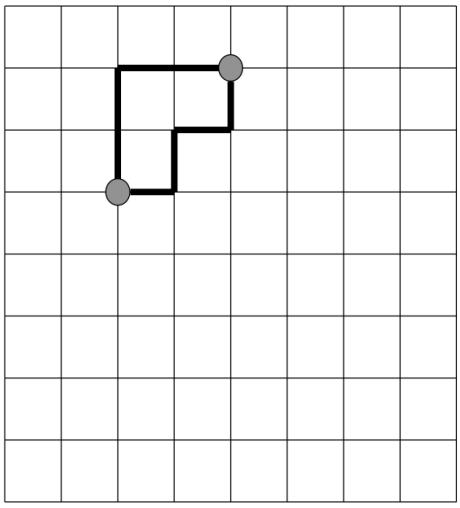

FIG. 1: The site errors here could have been caused by either error chain, however error chains with the same boundary in general have the same syndrome [2].

Note that this is only possible if the error chain does not wrap around the torus [2].

Because the error chain that generates the syndrome is somewhat arbitrary, we can choose error chains that give us an optimal syndrome for error correction. Given an error in the code, the probability for a set of errors c with the same effect on the code is (the starred items are from the dual lattice, not covered in this paper)

$$P(\bar{c}) = \sum_{b \in B_1} \sum_{b^* \in B_1^*} p_{c+b, c^*+b^*}, \tag{2}$$

where B_1 is the set of 1-chains that are the boundaries of 2-chains. The probability to obtain a given syndrome is therefore

$$P(\partial c) = \sum_{z^* \in H_1} \sum_{\bar{z^*} \in H_1^*} P(\bar{c} + \bar{z}\bar{c^*} + \bar{z^*}), \tag{3}$$

where $H_1 = Z_1/B_1$, and Z_1 is the set of all closed curves. The optimal method for error recovery is to choose a recovery chain with the highest conditional probability among those where the error and recovery chains are of the same homology class. This sets the success rate for

error correction to be

$$p_{max} = max_{\bar{z},\bar{z^*}\in H_1\times H_1^*}\frac{P(\bar{c}+\bar{z},\bar{c^*}+\bar{z^*})}{P(\partial c,\partial^* c^*)}. \tag{4}$$

By weighing each syndrome with its probability, we obtain the probability for successful error correction

$$\begin{aligned} p_{succ} &= \sum_{\partial c,\partial^* c^*} P(\partial c,\partial^* c^*)p_{max}(\partial c,\partial^* c^*) \\ &= \sum_{\partial c,\partial^* c^*} max_{\bar{z},\bar{z^*}\in H_1\times H_1^*}P(\bar{c}+\bar{z}\bar{c^*}+\bar{z^*}). \end{aligned} \tag{5}$$

One important property is that the success rate goes to 1 above the critical threshold, as explained in the following section [5].

RANDOM BOND ISING MODEL

We can learn more of the derivation of the error correction model by treating the system as a series of classical spins $\sigma_i = \pm 1$ located at each plaquette (equivalently located at the sites of the dual lattice) with probability p , where each site which has been mapped to a plaquette. The family of Hamiltonians for such a system is

$$H_\tau(s) = -\sum_{\langle ij\rangle}\tau_{ij}s_i s_j, \tag{6}$$

where $\tau_{ij} = \pm 1$ corrects for when the system is ferromagnetic or antiferromagnetic respectively and $\langle ij\rangle$ are nearest neighbor sites.

The partition function used to find the equilibrium state of the system is

$$Z(\beta,\tau) = \sum_s e^{\beta H_\tau(s)}. \tag{7}$$

Note that switching the spin of each plaquette would not change the energy. We can also preserve the energy if the spins are flipped in a nontrivial loop such as the bold region in figure 2 as long as we flip the signs of the interaction τ along the boundary. However, nothing can be done if in the same figure we flipped the signs along the bold line on the right which goes around the torus. This model is known as the random bond Ising model [5],[2].

With the random Bond Ising model, we can define the Nishimori line

$$e^{-2\beta} = \frac{p}{1-p} \tag{8}$$

in the phase diagram. The Nishimori line, when plotted against the phase diagram of the random bond Ising model in figure 3, gives the probability p_{crit} which is the threshold for error correction [5]. With a perfectly measured syndrome, this probability has been numerically determined to be approximately 10.9 percent [2].

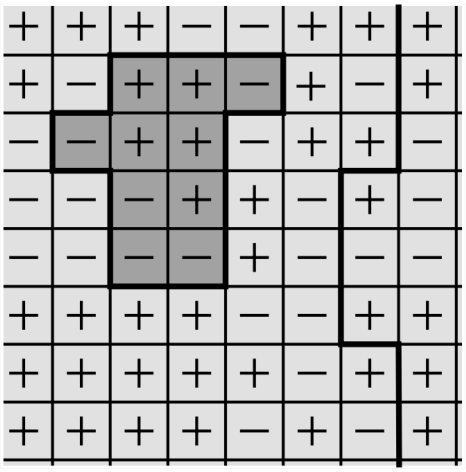

FIG. 2: The random bond ising model, where each plaquette has been assigned a random spin s_i with probability p. The energy can be preserved if all spins flip in the darkened nontrivial loop and their interactions with plaquettes outside of the loop reversed, but not for the bold line on the right [5].

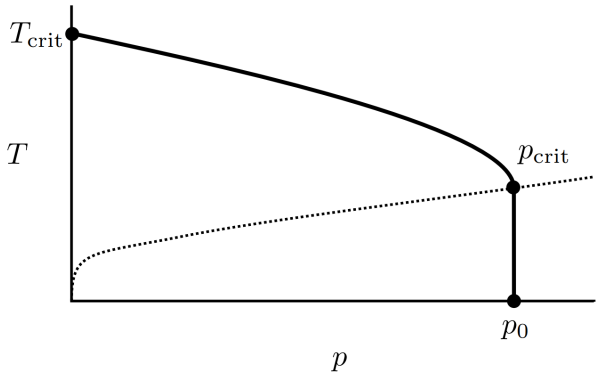

FIG. 3: The Nishimori line (dotted) plotted with the phase diagram of the antiferromagnetic random bond Ising mode. The intersection of the curves identifies the critical probability p_{crit} [5].

TORIC CODE AT T=0

As one could expect, surface codes are most stable at zero temperature. While coherence scales with system size, it is best to look at the thermodynamic limit with an arbitrarily large system size to generalize our results in code stability. Perturbations in the Hamiltonian due to thermal effects can change the dynamics of a system in a way that affects its ability to self-correct [6].

To be more specific on the effect of temperature, logical operators that are not part of the code's stabilizer $\hat{T}_\mu$ have a zero expectation value for long times in a local system

$$\langle\hat{T}_\mu\rangle_\alpha = 0. \tag{9}$$

We can find the autocorrelation time τ, which describes how long the system can continue self-correcting. Using the autocorrelator

$$G_{\hat{T}_\mu} = \langle \hat{T}_\mu(0)\hat{T}_\mu(t)\rangle, \tag{10}$$

which is valid at all positive temperatures, G goes to 0 as ($|t|/\tau$) goes to infinity, showing that autocorrelation time τ is finite. Only at zero temperature will the autocorrelation time be infinitely large, meaning that the code is indefinitely self-correcting. Source [7] demonstrates how this principle explicitly applies to the toric code, where $\hat{T}_\mu = X_\mu$ or Z_μ.

OTHER SURFACE CODES

Other models have been devised which present more information about a system. A widely-used version is the honeycomb lattice model, also by Kiatev. This model has 3 links to a site, and 3 distinct types of links, (x, y, and z) each facing in a different direction. This lattice gives the additional information of a spin at each site, with one of two choices corresponding to a spin 1/2 degree of freedom [4][7].

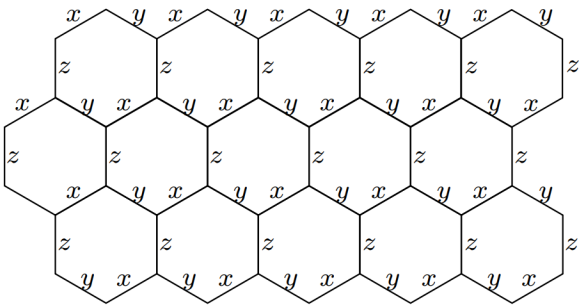

FIG. 4: The honeycomb planar code. Spins are found at each site, and there are 3 distinct types of links (x, y, and z) which are each at different angles on the diagram. [7]

Codes also exist that can describe higher-dimensional systems. The D=3 generalization of the 2D toric code has a site operator comprising of the 6 links that border a site, and a plaquette operator comprising of the 6 plaquettes that create a cube, with the same constraints as in the 2D case. [7]. 3D gates are more effective for local operators as each site has more neighbors [2].

CONCLUSION

The toric code is a surface code which has unique abilities in error correction. The random bond Ising model can be used to derive and analyze its error correcting properties. At zero temperature, the toric code has the ability to self-correct indefinitely. Many other surface codes similar to the toric code have found success by encoding additional information.

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