Summary: On the RG running of the entanglement entropy of a circle

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Defining the c-function to measure the degrees of freedom is a challenging problem. Casini and Huerta [1] proposed using the entanglement entropy of a circle to construct a c-function in 2+1 dimensions, based on the strong subadditivity and Lorentz covariance. However, in higher dimensions, divergent terms make it difficult to generalize this result. The microscopic structure of the surface in dimensions d¿2 contributes to different logarithmically divergent terms in the entropy, which may end up making the inequality of SSA less useful.

INTRODUCTION

C-theorem states that, in 1+1 dimensional spacetime, the degrees of freedom of a CFT at UV limit is higher than the one of the CFT at IR fixed point in the same RG flow trajectory. The degrees of freedom is measured by the c-function, but it's non-trivial to define such a quantity in higher dimensions. In 2+1 dim, with SSA and Lorentz covariance, Casini and Huerta proved the monotonicity of $c_0(r) \equiv S(r) - rS'(r)$ which can thus be defined as the c-function. In Casini and Huerta's paper, they used 3 chapters to discuss the 1+1 dim case, 2+1 dim case, and the general d+1 case, respectively.

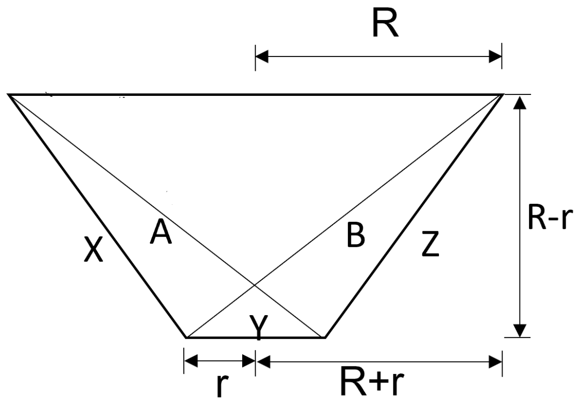

Figure 1:The interval A and B have radius $\sqrt{rR}$.

INTERVALS IN TWO-DIMENSIONAL SPACETIME

Consider a pair of boosted intervals A and B as shown in figure 1. By causality, all the information we have in XY should be kept in A as well, i.e. $S(XY) = S(A)$ and $S(YZ) = S(B)$. Now we can apply SSA

$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B) \qquad (1)$$

If we use the radius to rewrite the entropy, and also identify $S(A \cap B) = S(Y) = S(r)$ and $S(A \cup B) = S(XYZ) = S(R)$, we will have the inequality below

$$S(\sqrt{rR}) \geq S(R) + S(r) \qquad (2)$$

Here with Minkowski metric, we compute the radius of A and B

$$4r_A^2 = 4r_B^2 = (R+r)^2 - (R-r)^2 = 4rR \qquad (3)$$

Let $R = r + \epsilon$ and expand the two sides of the SSA inequality, then we will get $rS''(s) + S'(r) \leq$ If we further define the c-function $C(r) \equiv rS'(r)$, it's obvious that we have a c-function which is dimensionless and always decreasing $C'(r) \leq 0$

Moreover, this c-function can be linked with the one in Zamolodchikov's c-theorem. At the critical point $S(r) = \frac{c}{3}\log(\frac{r}{\epsilon}) + c_0$, where c is the Virasoro central charge, ϵ is a cutoff, and c_0 is a constant. Plug in this entropy to our c-function, $C(r) = \frac{c}{3}$, we get result consistent with Zamolodchikov's theorem which claims the central charge can be the c-function.
On the other hand, the other constant we have c_0 can not be defined as a c-function. That is because c_0 and its total variation are not universal, and they depend on the cutoff ϵ. However, there is an exception which happens when boundary conditions are imposed. In a fixed bulk CFT, c_0 is the boundary entropy. And the analogue of c-theorem is called the g-theorem.

CIRCLES IN 2+1 DIMENSIONS

The entropy of circles obtained by rotating a single circle around a point different from its center in a 2 dim plane diverges due to the corners of the intersection and the union of those rotated circles.

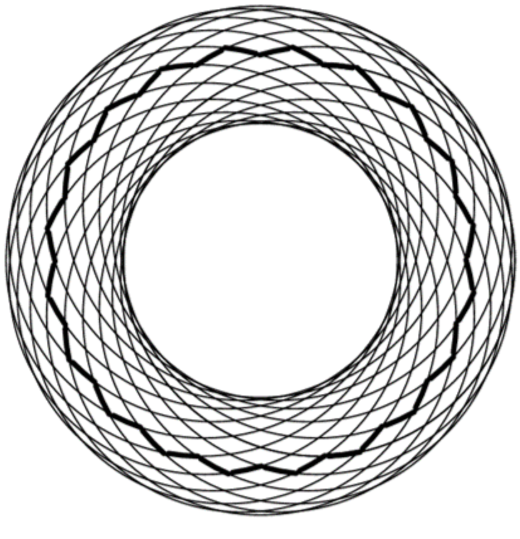

Figure 2:The rotated circles and the wiggly union of the corners

To remedy this divergence, we can boost the circle, and then rotate it. In such a way, all those wiggly corners approach circles on the light cone. And thus the entropy of any of them is finite.

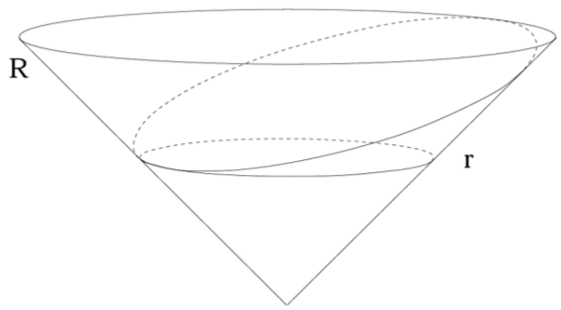

Figure 3:The rotated circles on the light cone

The boosted circle D lies on the plane $t = x\frac{R-r}{R+r} + \frac{2rR}{r+R}$. And the circle equation is $Rr = \left(x - \frac{R-r}{2}\right)^2 + y^2 - \left(t - \frac{R+r}{2}\right)^2$. Lastly, $y = x\tan(\frac{\theta}{2})$, where θ is the rotation angle. Using these equations, we can get the radius of the approximate circles

$$l = \sqrt{x^2+y^2} = \frac{2rR}{R+r-(R-r)\cos(\frac{\theta}{2})} \tag{4}$$

Next, we want to plug this in the SSA inequality. For N sets $\{X_i|i=1,2,...,N\}$, the SSA is

$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i \cap X_j)) + ... + S(\cap_i X_i) \tag{5}$$

The left hand side are the circles on the boosted spacial plane, and the right hand side are the approximate circles on the light cone.

$$NS(\sqrt{Rr}) \geq \sum_{i=1}^{N} S(\frac{2rR}{R+r-(R-r)\cos(\frac{\theta}{2})}) \tag{6}$$

After taking the $N \to \infty$ limit, the inequality becomes

$$S(\sqrt{Rr}) \geq \frac{1}{\pi}\int_0^{\pi} dz S(\frac{2rR}{R+r-(R-r)\cos(z)}) \tag{7}$$

Again, let $R = r + \epsilon$, then this inequality 7 will be turned into $S'' \leq 0$. Now consider a quantity $c_0(r) = S(r) - rS'(r)$, with the semi-negativity of S'', we also know $c_0' = -rS''(r) \geq 0$. This quantity c_0 is a good candidate of a c-function because it's dimensionless and monotonic. The tricky part here is that only the $\Delta c_0 = c_0^{ir} - c_0^{uv} \geq 0$ is well defined. The function c_0 itself can be sifted by arbitrary constant. This makes it hard to establish an exact correspondence between c_0 and physical properties. It's also the main difference between this 2+1 dim case and the original c-theorem.

SPHERES IN D+1 DIMENSIONS

In $d > 2$ dimensions, the union and intersection of the circles does not approach to a perfect sphere. The microscopic structure of the surface has additional contributions that remains to be studied.

If we naively directly generalize the result in 2+1 dimensions, we may obtain the SSA inequality (8) (neglecting the entropy of the microscopic structure)

$$S(\sqrt{Rr}) \geq \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d-1}{2})}\int_r^R dl \frac{(Rr)^{\frac{d-1}{2}}((l-r)(R-l))^{\frac{d-3}{2}}}{\sqrt{\pi}((R-r)/2)^{d-2}l^{d-1}} S(l) \tag{8}$$

Infinitesimal expansion $R = r + \epsilon$ will make it closer to the form we have before

$$rS''(r) - (d-2)S'(r) \leq 0 \tag{9}$$

For instance, we can substitute the conjectured form of the entanglement entropy in a 3+1-dimensional spacetime, $S(r) = c_2 r^2 + c_1 r + c_{\log}\log(R) + c_0$, into the inequality. This will give us

$$c_1 + 2\frac{c_{log}}{r} \geq 0 \tag{10}$$

Here's a contradiction, c_{log} has been proven negative in d=3, so the inequality does not hold for small r cases. The property of the leading divergent term is also interesting.

$$\Delta c_{d-1} \equiv c_{d-1}^{uv} - c_{d-1}^{ir} = -\int_0^{\infty} dr \Big(\frac{S'(r)}{(d-1)r^{d-2}}\Big)' \geq 0 \tag{11}$$

In the case of a massive scalar theory,

1. $\Delta c_{d-1} = \gamma_d vol(s^{d-1}) m^{d-1} \log(m\epsilon)$ for d odd
   where $\gamma_d = (-1)^{\frac{d-1}{2}} \left[6(4\pi)^{\frac{d-1}{2}}((d-1)/2)!\right]^{-1}$
2. $\Delta c_{d-1} = \gamma_d vol(s^{d-1}) m^{d-1}$ for d even
   and $\gamma_d = (-1)^{\frac{d}{2}+1} \left[12(2\pi)^{\frac{d-2}{2}}((d-1)/2)!!\right]^{-1}$

The sign depends on the dimension.

[1] H Casini and Marina Huerta. Renormalization group running of the entanglement entropy of a circle. *Physical Review D*, 85(12):125016, 2012.