Effects of local decoherence on topological order

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We discuss the effect of local decoherence on topological order based on the recent papers [1, 2]. By studying 2D toric code with bit-flip errors as a concrete example, we show that two information-theoretic quantities undergo a transition at the same error rate. An effective filed theory describing the general decoherence-induced transitions is also presented.

INTRODUCTION

As discussed in the lecture, a system is said to exhibit topological order if different degenerate ground states cannot be distinguished by any local operator. This definition relies on the assumption that the system is always in a pure state. However, one motivation of realizing toplological orders is that they can protect quantum memories under decoherence, which necessarily drives the system to the mixed states. This naturally raises several questions: How can topological order be quantified in mixed states? How stable are the topological orders under decoherence? Is there any error-induced singularity in the mixed state?

In this note, we address the above questions based on the recent papers [1, 2]. We begin by studying 2D toric code with bit-flip errors as a concrete example of the topological order under local decoherence. We show that the trace of the n-th moment of the corrupted density matrix can be mapped to the statistical models, allowing the identification of the phase transition point. Two information-theoretic diagnostics are then introduced and related to the observables of the statistical models. After having some intuition, we propose to use the error-field double state as the general formalism and develop an effective field theory to characterize the decoherence-induced transitions.

TORIC CODE UNDER BIT-FLIP AND PHASE ERRORS

We consider 2D toric code on the square lattice with periodic boundary condition. The ground state subspace is stabilized by the Hamiltonian

$$H_{\mathrm{TC}} = -\sum_s A_s - \sum_p B_p, \tag{1}$$

where $A_s = \prod_{l\in s} X_l$ and $B_p = \prod_{l\in p} B_p$ are operators associated with vertices and plaquettes, respectively. Noting the four-fold degeneracy of the toric code on the torus, the maximally mixed density matrix in the code

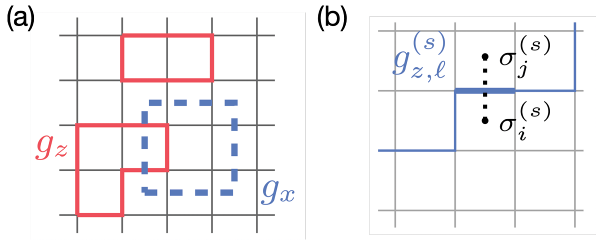

FIG. 1: (a) The density matrix of the toric code ground state can be written as the equal weight superposition of the loop operators g_x and g_z. (b) Regarding the loop configurations $g_z^{(s)}$ as the domain walls of Ising spins, one can map the n-th moment of the density matrix to the partition function of $(n-1)$-flavor Ising spins.

space can then be written as

$$\rho_0 = \frac{1}{4}\prod_s \frac{1+A_s}{2}\prod_p \frac{1+B_p}{2} \tag{2}$$

$$= \frac{1}{2^N}\sum_{g_z} g_z \sum_{g_x} g_x, \tag{3}$$

where $g_z(g_x)$ denotes $Z(X)$ loop on the lattice(dual lattice). Eq.(3) suggests an interpretation of ρ_0 as the equal weight superposition of two kinds of loop operators g_x and g_z [Fig.1(a)]. Now, consider subjecting the system to the bit-flip channel

$$\rho = \mathcal{N}_X[\rho_0] = (\prod_i \mathcal{N}_{X,i})[\rho_0], \tag{4}$$

where

$$\mathcal{N}_{X,i}[\rho] = (1-p)\rho + pX_i\rho X_i, \tag{5}$$

describes the local bit-flip error. It then follows that the corrupted density matrix becomes

$$\rho = \frac{1}{2^N}\sum_{g_x} g_x \sum_{g_z} e^{-\mu_z |g_z|} g_z, \tag{6}$$

where $|g_z|$ denotes the length of the loop and $\mu_z = -\log(1-2p)$. Therefore, the effect of the decoherence corresponds to adding length tension to the loop configuration.

Does the decoherence channel destroy the topological order? If so, how do we quantify it? Using the purification trick, one can regard the corrupted mixed states as a pure state related to the topologically ordered state by a depth-1 unitary circuit: $|\Phi\rangle = \prod_i U_i(p)|\Psi_0\rangle \otimes |0\rangle_i$ [Fig.2]. Therefore, any linear functions of the density matrix, e.g., any physical observable, must be a smooth function of the error rate. However, it doesn't preclude the possibility of using non-linear functions of ρ to probe the transitions. In the following, we consider the n-th moment of the density matrix $\mathrm{tr}(\rho^n)$ and demonstrate that it can be mapped to the partition function of statistical models of $(n-1)$-flavor Ising spins.

Using the property that the trace of the Pauli matrices vanish, one can show that

$$\mathrm{tr}\rho^n = \frac{1}{2^{(n-1)N}} \mathcal{Z}_n, \tag{7}$$

where $\mathcal{Z}_n = \sum_{\{g_z^{(s)}\}} e^{-H_n}$ with

$$H_n = \mu_z \Big(\sum_{s=1}^{n-1} |g_z^{(s)}| + |\prod_{s=1}^{n-1} g_z^{(s)}| \Big). \tag{8}$$

The equivalence between $\mathrm{tr}\rho^n$ and statistical models is manifested by regarding the loop configurations $g_\alpha^{(s)}$ as the domain walls of the s-th flavor Ising spins:

$$|g_{z,l}^s| = \frac{1 - \sigma_i^{(s)}\sigma_j^{(s)}}{2}, \tag{9}$$

where l labels the link and i, j are the connected sites dual to l [Fig.1(b)]. H_n can then be written as the effective Hamiltonian of the $(n-1)$-flavor Ising spins

$$H_n = -\frac{\mu_z}{2} \sum_{\langle i,j, \rangle} \Big(\sum_{s=1}^{n-1} \sigma_i^{(s)}\sigma_j^{(s)} + \prod_{s=1}^{n-1} \sigma_i^{(s)}\sigma_j^{(s)} \Big). \tag{10}$$

Here the first term is the usual ferromagnetic interaction of the individual Ising spins while the second term describes to the coupling among different flavors of the Ising spins. This model possesses a global symmetry $G^{(n)} = (\mathbb{Z}_2^{\otimes n} \ltimes \mathcal{S}_n)/\mathbb{Z}_2$, where S_n is the permutation symmetry over n ising spins. One can show that increasing μ_z (equivalent to increasing the error rate p) drive the system from paramagnetic phase to ferromagnetic phase which completely breaks the $G^{(n)}$ symmetry.

INFORMATION THEORETIC DIAGNOSTICS

To have more physical understanding of the transitions, we consider two information theoretic diagnostics. The first diagnostic is motivated by the fact that applying $w_\alpha(\mathcal{P})$, an open string operator that creates an anyon pair $\alpha\alpha'$ at the opposite ends of the path $\mathcal{P}$, to the system creates an orthogonal ground state in the absence of decoherence. One can then test whether ρ_0

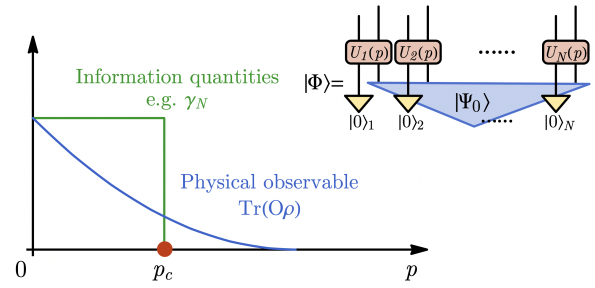

FIG. 2: The corrupted mixed states is a pure state related to the topologically ordered state by a depth-1 unitary circuit. Therefore, only information quantities which are non-linear functions of the density matrix can probe the transition.

and $\rho_{0,\alpha} := w_\alpha(\mathcal{P})\rho_0 w_\alpha(\mathcal{P})^\dagger$ are still distinguishable after subjecting to the decoherence. This can be quantified by the Renyi relative entropy:

$$D^{(n)}(\rho||\rho_\alpha) := \frac{1}{1-n} \log \frac{\mathrm{tr}\rho\rho_\alpha^{n-1}}{\mathrm{tr}\rho^n}. \tag{11}$$

It is expected that $D^{(n)}(\rho||\rho_\alpha)$ diverges below the critical error rate while saturates to a constant above the error rate.

The second diagnostic is the Renyi negativity of even order:

$$\mathcal{E}_A^{(2n)}(\rho) := \frac{1}{2-2n} \log \frac{\mathrm{tr}\rho^{T_A}}{\mathrm{tr}\rho^{2n}}, \tag{12}$$

where T_A denotes the partial transpose of the subsystem A. It is conjectured that $\mathcal{E}_A^{(2n)} = c|\partial A| - \gamma_N$, where $\gamma_N = \log 2$ is the topological entanglement negativity characterizing the mixed-state long-range entanglement.

Using the mapping discueed in previous section, one can show that the relative Renyi entropy is related to order parameter n-th flavor Ising spins:

$$D^n(\rho||\rho_\alpha) = \frac{1}{1-n} \log\langle \sigma_{i_l}^{(1)} \sigma_{i_r}^{(1)} \rangle, \tag{13}$$

where $\sigma_j^{(1)}$ is the first flavor of the Ising spin at site j.

On the other hand, the entanglement Renyi negativity is related to the excess free energy attributed to a single flavor of Ising spins on the boundary ∂A in the same direction ΔF_A :

$$\mathcal{E} = \Delta F_A. \tag{14}$$

In the paramagnetic phase, the spins aligned across the boundary can fluctate together, leading to the sub-leading term in the excess free energy $\log 2$. On the other hand, in the ferromagnetic phase, the aligned boundary spins are fixed by the global magnetization, leading to the vanishing sub-leading term.

In short, both Renyi relative entropy and Renyi entanglement negativity can be mapped to the physical observables of the n-th flavor Ising model, and thus detect the

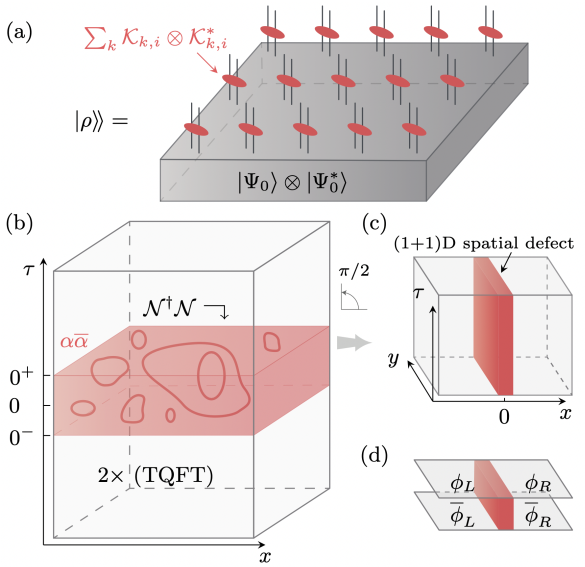

FIG. 3: (a) Treating the density matrix as a state vector in the double Hilbert space, the error-corrupted density matrix is equivalent to the non unitary operation on the two copies of topologically ordered pure states. (b) The double state $|\rho_0\rangle\rangle(\langle\langle\rho_0||)$ can be described by the two copies of TQFT in the Euclidean half spacetime $\tau < 0 (\tau > 0)$, and the effect of decoherence corresponds to the temporal defect at $\tau = 0$. (c) Performing a $\pi/2$ rotation maps the temporal interface at $\tau = 0$ onto the spatial interface at $x = 0$. (d) The path integral describes a double 2D topological order with a 1D defect.

same transition point. Their consistency strongly suggests that there is a breakdown of topological orders at a finite error rate.

FIELD THEORY DESCRIPTION: ERROR-FIELD DOUBLE STATE AND THE MAPPING TO (1+1) D BOUNDARY PHASES

To understand the general effect of local decoherence without referring to microscopic details, we now employ the effective field theory description.

We begin by noting an alternative way to understand the effect of decoherence: treating the density matrix as a state vector in the double Hilbert space using the Choi–Jamiołkowski (CJ) isomorphism:

$$\rho_0 = |\Psi_0\rangle\langle\Psi_0| \to |\rho\rangle\rangle = |\Psi_0\rangle \otimes |\Psi_0^*\rangle, \tag{15}$$

the error-corrupted density matrix is equivalent to the non-unitary evolution on the two copies of topologically ordered pure states:

$$|\rho\rangle\rangle = \mathcal{N}(|\Psi_0\rangle \otimes |\Psi_0^*\rangle) \tag{16}$$

$$= \prod_l \sum_k \mathcal{K}_{k,l} \otimes \bar{\mathcal{K}}_{k,l}(|\Psi_0\rangle \otimes |\Psi_0^*\rangle), \tag{17}$$

where $\mathcal{K}_{k,l} \otimes \bar{\mathcal{K}}_{k,l}$ denotes the CJ transfromed Kraus operator [see Fig.3(a)]. The advantage of this perspective lies in the fact that we can now treat $\mathcal{K}_{k,l} \otimes \bar{\mathcal{K}}_{k,l}$ as an operator creating two neighboring anyon pairs $(k\bar{k})_i, (k\bar{k})_j$ from the two copies of topollogically ordered ground states. For example, the bit-flip error discussed in previous section has two CJ-transformed Kraus operators: $\mathcal{K}_{0,l} \otimes \bar{\mathcal{K}}_{0,l} = (1-p)\mathbb{1}_l \otimes \bar{\mathbb{1}}_l$ and $\mathcal{K}_{1,l} \otimes \bar{\mathcal{K}}_{1,l} = pX_l \otimes \bar{X}_l$. The $k = 1$ case corresponds to creating two neighrboring flux anyon pairs $(m\bar{m})_i, (m\bar{m})_j$. At a sufficiently large error rate p, it is expected that the system will be driven from two copies of topologically ordered phase to other phases through anyon condensation.

For simplicity, we focus on the 2nd moment of the density matrix $\mathrm{tr}\rho^2$ and consider Abelian topological orders with incoherent errors. In the double state formalism, $\mathrm{tr}\rho^2$ corresponds to the normalization of $|\rho\rangle\rangle$:

$$\mathrm{tr}\rho^2 = \langle\langle\rho|\rho\rangle\rangle = \langle\langle\rho_0|\mathcal{N}\mathcal{N}|\rho_0\rangle\rangle. \tag{18}$$

The double state $|\rho_0\rangle\rangle(\langle\langle\rho_0||)$ can then be described as the two copies of TQFT in the Euclidean half spacetime $\tau < 0$ $(\tau > 0)$, and the effect of decoherence corresponds to the temporal defect at $\tau = 0$ [Fig.3(b)].

Since the temporal defects of toplogical quantum field theory (TQFT) are complicated to analyze and largely unexplored, it is more convenient to perform a $\pi/2$ rotation to map the temporal interface at $\tau = 0$ onto the spatial interface at $x = 0$. The path integral then describes a $(2+1)D$ system with $(1+1)D$ spatial defect [Fig.3(c)]. The spatial defect is much easier to analyze, as the boundary of TQFT has a simple descriptions in terms of compact bosons. Denoting $s = L(R)$ the EFD originated from $|\rho_0\rangle\rangle(\langle\langle\rho_0|)$ and $\phi_s(\bar{\phi}_s)$ the compact bosons at the ket(bra) Hilbert space, the low-energy physics of the four decoupled edge modes $\phi := [\bar{\phi}_R, \phi_R, \phi_L, \bar{\phi}_L]$ can be described by the Lagrangian

$$\mathcal{L}_0 = \frac{1}{4\pi}\sum_{I,J} \mathbb{K}^{(2)}_{IJ} i\partial_\tau\phi^I\partial_y\phi^J - \mathbb{V}^{(2)}_{IJ}\partial_y\phi^I\partial_y\phi^J, \tag{19}$$

Here $\mathbb{K}^{(2)} = K \oplus (-K) \oplus K \oplus (-K)$ with K an integer-valued matrix characterizing toppological orders and $\mathbb{V}^{(2)} = V \oplus V \oplus V \oplus V$ a non-universal positive definite matrix. However, note that without the decoherence, $\phi_L(\bar{\phi}_L)$ and $\phi_R(\bar{\phi}_R)$ are strongly coupled. Therefore, one should consider another Lagrangian which is present even in the absence of the decoherence:

$$\mathcal{L}_1 = \sum_\Lambda B_\Lambda\Big[\cos\Big(\sum_I (K\Lambda)_I(\phi_L^I + \phi_R)^I\Big) + (\phi_s \leftrightarrow \bar{\phi}_s)\Big]. \tag{20}$$

Here, Λ is an M-component integer vector. When B_Λ dominates (corresponding to no decoherence), the bosonics fields $\phi_L + \phi_R$, $\bar{\phi}_L + \bar{\phi}_R$ are pinned by the cosine terms.

On the other hand, the effect of decoherence only couples the fields from the same copy of density matrix, which can be described by the Lagrangian

$$\mathcal{L}_N = \sum_{\Lambda} C_\Lambda \sum_{s=L,R} \cos\Big(\sum_{I=1}^{M} (K\Lambda)_I (\phi_s^I - \bar{\phi}_s^I)\Big). \qquad (21)$$

Taking all the aforementioned terms together, the total Lagrangian is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_{\mathcal{N}}, \qquad (22)$$

and our goal is to classify all the possible gapped phases. Using the framewrok developed in Ref.[3] and the symmetry contraint enforced by the EFD, the possible edge phases for toric code mdouble semion model, and the $\nu = 1/3$ Laughlin state are summarized in Fig.4.

DISCUSSION

We have studied the effect of local decoherence on topological order We show that there is a error-induced singularity in the mixed state which can be identified by the information theoretic qantities. A universal description characterizing the impace of decoherence is also presented. It will be interesting to generalize the discussion to the quantum channels including incoherent error, non-Abelian topological orders, and even chiral topoligical orders.

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[2] R. Fan, Y. Bao, E. Altman, and A. Vishwanath, *Diagnostics of mixed-state topological order and breakdown of quantum memory* (2023), URL `https://arxiv.org/abs/2301.05689`.

[3] M. Barkeshli, C.-M. Jian, and X.-L. Qi, Phys. Rev. B **88**, 235103 (2013), URL `https://link.aps.org/doi/10.1103/PhysRevB.88.235103`.

Model	Memory	Edge condensate (generators of Lagrangian subgroup)		Error that realizes the phase
	Quantum	I	$e_L e_R,\ \bar{e}_L \bar{e}_R,\ m_L m_R,\ \overline{m}_L \overline{m}_R$	No error
Toric code	Classical	II	$e_L \bar{e}_L,\ e_R \bar{e}_R,\ e_L \bar{e}_R,\ m_L \overline{m}_L m_R \overline{m}_R$	Incoherent e error
$K_{\mathrm{TC}} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	Classical	III	$m_L \overline{m}_L,\ m_R \overline{m}_R,\ m_L \overline{m}_R,\ e_L \bar{e}_L e_R \bar{e}_R$	Incoherent m error
	Classical	IV	$f_L \overline{f}_L,\ f_R \overline{f}_R,\ f_L \overline{f}_R,\ e_L \bar{e}_L e_R \bar{e}_R$	Incoherent f error
	Trivial	V	$e_L \bar{e}_L,\ e_R \bar{e}_R,\ m_L \overline{m}_L,\ m_R \overline{m}_R$	Any two types of incoherent errors
	Quantum	I	$m_{aL} m_{aR},\ \overline{m}_{aL} \overline{m}_{aR},\ m_{bL} m_{bR},\ \overline{m}_{bL} \overline{m}_{bR}$	No error
Double semion	Quantum	II	$m_{aL} \overline{m}_{aL},\ m_{aR} \overline{m}_{aR},\ m_{bL} m_{bR},\ \overline{m}_{bL} \overline{m}_{bR}$	Incoherent m_a error
$K_{\mathrm{DS}} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$	Quantum	III	$m_{bL} \overline{m}_{bL},\ m_{bR} \overline{m}_{bR},\ m_{aL} m_{aR},\ \overline{m}_{aL} \overline{m}_{aR}$	Incoherent m_b error
	Quantum	IV	$b_L \bar{b}_L,\ b_R \bar{b}_R,\ b_L b_R,\ m_{aL} \overline{m}_{aL} m_{aR} \overline{m}_{aR}$	Incoherent b error
	Trivial	V	$m_{aL} \overline{m}_{aL},\ m_{aR} \overline{m}_{aR},\ m_{bL} \overline{m}_{bL},\ m_{bR} \overline{m}_{bR}$	Any two types of incoherent errors
$\nu = 1/3$ Laughlin state	Quantum	I	$\eta_L \eta_R^2, \bar{\eta}_L \bar{\eta}_R^2$	No error
$K_{1/3} = (3)$	Trivial	II	$\eta_L \bar{\eta}_L, \eta_R \bar{\eta}_R$	Incoherent error for quasiparticles

FIG. 4: Decoherence-induced phases in the Toric code, double semion model, and $\nu = 1/3$ Laughlin state subject to incoherent errors.