What is ... Entanglement Bootstrap?

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There will be an extended version of this final paper.

INTRODUCTION

There are many facets of the entanglement bootstrap (EB) program. It can be seen as a collection of physics axioms that classify the phases of matter. It can be regarded as a fundamental mathematical inquiry into the quantum local-global principle. It can be viewed as a novel numerical method that complements quantum many body bootstrap. It can also be seen a philosophical question about the epistemology of the universal wave function.

Instead of covering all these aspects, we will focus ourselves to understand the physical motivations of EB, then present a set of axioms with the desired properties.

TWO GUIDING PRINCIPLES OF EB – LOCAL STATES AND LOCAL-GLOBAL PRINCIPLE

There are two aspects in EB that are different from the traditional way of doing physics: 1. One is the focus on the ground state and its local reduced density matrices. 2. The second is the focus on the local-global principle, which is to infer the global properties, such as the ground state and topological invariants, from the local reduced density matrices. Together, the goal is to argue that the local reduced density matrices of a ground state contains all the information you want to know in physics.

Why states?

When you go to a physics class, the professor often begins by writing down a Lagrangian or a Hamiltonian. In EB, however, we start from the state. Why so?

The main reason is that a state is simpler and more explicit than a Hamiltonian. When handed a Hamiltonian, the first thing to do is often to find its ground state. If that is the case, why not start with the ground state?

Why local states?

Instead of studying the global wave function, EB focuses on the local reduced density matrices. The reason is that a local state is simpler to describe than a global state which also reflects the locality of physics.

A general quantum state over n qubits requires roughly 2^n complex numbers to describe. However, the majority of the states are not physical and do not belong to the ground state of a local Hamiltonian. Instead, if we consider the local reduced density matrices induced by some global wave function it takes only $\Theta(n)$ (linear to n) many complex numbers. Furthermore, it can be shown that the maximal entropy state consistent with these local states is the Gibbs state of a local Hamiltonian. Thus, using local states costs less parameters and recovers locality.

Why local-global principle?

Despite these advantages, it would be unfavorable if these simplifications lose certain physical information. In particular, it would be nice to show: 1. The Hamiltonian can be recovered from the ground state. 2. The ground state can be recovered from the local reduced density matrices. I believe that both statements are true. In the following discussion, we will focus on the second statement which is commonly refer to as the local-global principle in mathematics. (Caveat: I'm using the term local-global principle more casually than a mathematician would.)

LOCAL-GLOBAL PRINCIPLE – IDENTITY AND GLUABILITY AXIOMS

To discuss local-global principle for quantum states, we first review the local-global principle for functions on a manifold. As we will see, functions on a manifold satisfy two important properties called the identity and gluability axioms.

Recall a (real) function f on a manifold M is described by the value $f(x) \in \mathbb{R}$ at each point $x \in M$ and each subset $U \subseteq M$ leads to a local function $f|_{U_i}$. Later f will play the role of the global wave function ρ and $f|_{U_i}$ will play the role of the reduced density matrix $\rho|_{U_i} = \mathrm{tr}_{\overline{U_i}}\, \rho$ where $\overline{U_i}$ is the complement of U_i. We denote the set of functions over a set U as $\mathcal{F}(U) \coloneqq \{f : U \to \mathbb{R}\}$.

A set of subsets $\{U_i\}$ is a cover of U if $\bigcup_i U_i = U$. It is clear that a cover induces the map

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i),$$

where $f \mapsto (i \mapsto f|_{U_i})$.

Given such a map one often ask for two properties:

- Identity axiom or Uniqueness: Given $f_1, f_2 \in \mathcal{F}(U)$ and $f_1|_{U_i} = f_2|_{U_i}$ for all i, then $f_1 = f_2$.

- Gluability axiom or Existence: Given $f_i \in \mathcal{F}(U_i)$ for all i, such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, then there is some $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$ for all i.

It is not hard to show that functions satisfy the two axioms. For identity, because $\{U_i\}$ forms a cover of U, each point $x \in U$ belongs to some subset $x \in U_i$. Since $f_1|_{U_i} = f_2|_{U_i}$, we have $f_1(x) = f_2(x)$. Therefore, $f_1 = f_2$ and the identity axiom holds. For gluability, we find f by defining $f(x)$ for each $x \in U$. Let $j(x)$ be the index whose subset covers x, $x \in U_{j(x)}$. We set $f(x) = f_{j(x)}(x)$ which defines f. To check $f|_{U_i} = f_i$, we suffice to check $f(x) = f_i(x)$ for all $x \in U_i$. This holds because $f(x) = f_{j(x)}(x) = f_i(x)$ where the first equality follows from the definition of f and the second equality follows from $f_{j(x)}|_{U_{j(x)} \cap U_i} = f_i|_{U_{j(x)} \cap U_i}$.

We now learned that the identity and gluability axioms are satisfied by functions. More generally, the two axioms hold for a large family of mathematical objects. The framework to discuss these properties is known as sheaf theory, where you can learn more from [1, Ch 2].

IDENTITY AND GLUABILITY AXIOMS FAIL FOR QUANTUM STATES

Given the generality of the identity and gluability axioms, we want to apply them to quantum states. Unfortunately, both axioms fails to hold.

We first discuss how the identity axiom fails for quantum states. To demonstrate its failure, we need to find distinct global quantum states with identical local reduced density matrices. One example comes from EPR states. Consider four distinct states over AB, $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. It is clear that they have identical reduced density matrices $\rho_A = \rho_B = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$. The problem arises because the reduced density matrices are mixed, which permits the existence of correlations.

Another example is the toric code on a torus. We know that toric code has four orthogonal ground states, yet all of them look locally identical. In fact, it is this local indistinguishability that gives toric code the ability to perform error correction. More generally, any topological quantum field theory with degenerate ground states fails to satisfy the identity axiom.

We now turn out attention to the gluability axiom. To demonstrate its failure, we need to find local reduced density matrices that agree at the overlap, but cannot be induced from a global quantum state. One example comes from quantum monogamy which says that it is impossible to have a global state ρ_{ABC} with ρ_{AB} and ρ_{BC} be both EPR pairs. More precisely, suppose we have three regions, A, B, C, each contains a qubit, and covered by two charts AB, BC. The local reduced density matrices are $\rho_{AB} = |EPR\rangle_{AB}\langle EPR|_{AB}$ and $\rho_{BC} = |EPR\rangle_{BC}\langle EPR|_{BC}$ where $|EPR\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. It is clear that the two density matrices agree at B with $\rho_B = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$. However, there is no global density matrix ρ_{ABC} such that $\mathrm{tr}_C(\rho_{ABC}) = \rho_{AB}$ and $\mathrm{tr}_A(\rho_{ABC}) = \rho_{BC}$. To see why ρ_{ABC} cannot exist, we utilize strong subadditivity (SSA). If ρ_{ABC} exists, SSA implies $S(AB) + S(BC) \geq S(ABC) + S(B)$. Now, $LHS = 0$ but $RHS \geq \log 2$, which leads to a contradiction. To summarize, even though ρ_{AB} and ρ_{BC} agree at the overlap, they do not glue into a consistent global wave function.

These counterexamples may seem like a big hit to the philosophy of EB. Nevertheless, there are reasons to remain hopeful. In the next section, we impose two additional conditions to the local reduced density matrices. With the new conditions, the theory now satisfies the identity and gluability axioms, with a few modifications.

RECOVER IDENTITY AND GLUABILITY AXIOMS THROUGH TWO CONDITIONS

The two additional conditions are called **A0** and **A1** first proposed in [2, 3], then utilized in 2+1d TQFT to explain the emergence of anyons and fusion rules in [4]. The following presentation of **A0** and **A1** will be a sketch of the ideas instead of being complete or rigorous. The details will be provided in the extended version.

A0, which I called local purifiability, says that the local reduced density matrix satisfies

$$S(\rho_{BC}) + S(\rho_C) = S(\rho_B)$$

for the partition topologically equivalent to the left figure.

A1, which I called extendability, says that the local reduced density matrix satisfies

$$S(\rho_{BC}) + S(\rho_{CD}) = S(\rho_B) + S(\rho_D)$$

for the partition topologically equivalent to the right figure.

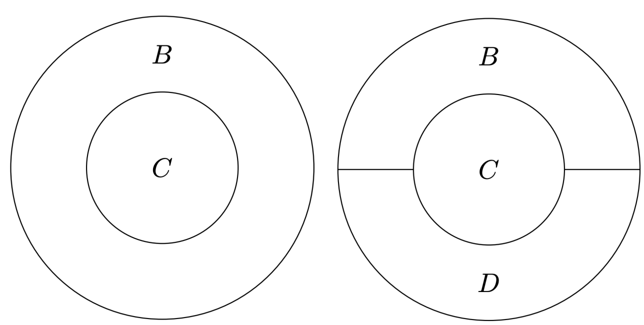

FIG. 1: Left: the partition for **A0**. Right: the partition for **A1**.

It is worth noting that the two equalities saturate two well-known inequalities in quantum information, subadditivity $S(A)+S(C) \geq S(AC) \Leftrightarrow S(BC)+S(C) \geq S(B)$, and strong subadditivity $S(BC) + S(AB) \geq S(B) + S(ABC) \Leftrightarrow S(BC) + S(CD) \geq S(B) + S(D)$. We denote A as the environment in the purification of the local reduced density matrix and utilize $S(X) = S(\overline{X})$ for a pure state. The saturation of inequalities impose significant constraints on the state. This is illustrated by the following key features of **A0** and **A1** which also explains the origin of their names.

A0 implies ρ_{BC} can be written as

$$\rho_{BC} = W_B(\rho_{B_1} \otimes |\psi_{B_2C}\rangle\langle\psi_{B_2C}|)W_B^\dagger$$

where $W_B \colon \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2} \to \mathcal{H}_B$ is an embedding. The appearance of $|\psi_{B_2C}\rangle$ intuitively means the uncertainty in C can be completely purified by B, hence **A0** is called local purifiability.

A1 together with SSA imply $I(A : C|B) = 0$ where A is any subset in the complement of BCD. Now, because ρ_{ABC} forms a Markov chain, we can reconstruct ρ_{ABC} from ρ_{AB} and ρ_{BC} by writing $\rho_{ABC} = \rho_{AB}\rho_B^{-1}\rho_{BC}$. This extends ρ_{AB} to ρ_{ABC}, thus **A1** is called extendability.

We now state the modified identity and gluability axioms that hold for quantum states with **A0** and **A1**. We denote the set of density matrix over a set U as $\mathcal{D}(U)$.

Theorem 1 (Modified identity axiom) *When U is simply connected, given $\rho^1, \rho^2 \in \mathcal{D}(U)$ and $\rho^1|_{U_i} = \rho^2|_{U_i}$ that satisfies **A0** and **A1** for all i, then $\rho^1|_{U_{\text{int}}} = \rho^2|_{U_{\text{int}}}$, where U_{int} is a subset of U with nonzero distance away from the boundary.*

Theorem 2 (Modified gluability axiom) *Given $\rho_i \in \mathcal{D}(U_i)$ that satisfies **A0**, **A1** for all i, such that $\rho_i|_{U_i \cap U_j} = \rho_j|_{U_i \cap U_j}$ for all i, j, then there is some $\rho \in \mathcal{D}(U)$ such that $\rho|_{U_i} = \rho_i$ for all i.*

The modified identity and gluability axiom are related but not equivalent to Prop 3.5 and Lemma 3.8 in [4]. Here, we provide the intuition of the proof. For the gluability axiom, **A1** and the Markov property allow us to merge the local states together into the global state. For the identity axiom, **A0** implies local region is purified. Because pure state does not allow further correlations (recall the counterexample of EPR states) this gives the identity axiom.

CONCLUSION

There are many foundational open questions in EB. 1. One is to extend the two conditions to all quantum phases of matter. For example, the case for 1+1d CFTs is explored in [5]. 2. Another is to apply the conditions to study quantum phases of matter. For example, the emergence of anyons and fusion rules in 2+1d TQFTs is studied in [4]. 3. The last and perhaps the most important is to understand the robustness of identity and guability, i.e., does the identity and guability axioms hold when the local density matrices violates **A0** and **A1** slightly? The power of this framework is only beginning to be explored.

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