Detection of Topological Order in Gapped Quantum Spin Liquids

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This paper constitutes an overview of the study of gapped quantum spin liquids using their entanglement properties. The detection of topological order in candidate wave functions by using the topological entanglement entropy is treated. This can be used to identify quantum spin liquid ground states. A concrete example is given for a chiral spin liquid on the torus, for which the topological entanglement entropy is computed with a Variational Monte Carlo algorithm.

I. INTRODUCTION

Quantum spin liquids (QSLs) [1] are intriguing phases of strongly correlated spins that do not magnetically order even at zero temperature. They possess various exotic properties, such as fractionalized excitations, long-ranged quantum entanglement and are not described by a local order parameter. Quantum spin liquids can be gapped or gapless. The former category of QSLs also has the amazing property of hosting topological order (TO), a robust ground state degeneracy that depends on the topology of the manifold on which the system lives.

Due to their strongly interactive nature, QSLs are quite hard to study. Given a wave function, a challenging task is to determine if it describes the ground state of a QSL. However, it turns out that the entanglement can be of great use for answering this question.

The goal of this short paper is to illustrate how the topological entanglement entropy (TEE) can be used to detect topological order in trial wave functions, allowing to identify gapped quantum spin liquids [2]. The paper is structured as follows: Section II presents a brief review of the topological entanglement entropy, while Section III illustrates how to build a QSL trial wave function, specifically for a chiral QSL. Section IV shows how to extract the TEE and to compute it via a Variational Monte Carlo algorithm, while we conclude in Section V.

II. TOPOLOGICAL ENTANGLEMENT ENTROPY

Consider a two-dimensional gapped system partitioned in two subregions A and B, with a contractible boundary. The amount of quantum entanglement between the two subsystems A and B can be quantified by the Renyi entropies (of region A), defined as

$$S_n(A) = \frac{1}{1-n} \log \left(\mathrm{Tr}\, \rho_A^n \right), \tag{1}$$

where ρ_A is obtained by tracing out the degrees of freedom of region B. The Renyi entropies respect the area law [3]

$$S_n(A) = a_n L - b_0 \gamma + \mathcal{O}(1/L), \tag{2}$$

where L is the boundary length between A and B, a_n is a n-dependent non-universal coefficient and b_0 is the number of disconnected pieces of the boundary.

The coefficient γ is called the topological entanglement entropy (TEE) [4, 5] and is a robust universal property of the ground state wave function. It is only nonzero for phases with topological order. It has been shown that $\gamma = \log D$, with $D = \sqrt{\sum_a d_a^2}$, the total quantum dimension of the topological phase. The sum over a runs over all the different types of anyons, while d_a is the quantum dimension of anyon a. This name comes from the fact that for a large number M of anyons of type a, the Hilbert space dimension goes as d_a^{M-2} [6]. Note that for abelian anyons, $d_a = 1$, while for non-abelian anyons, $d_a > 1$.

Therefore, we see that the TEE is a powerful tool to identify gapped QSLs. Indeed, by computing the TEE from a ground state wave function, we immediately know if the associated phase has topological order and is thus a valid gapped QSL candidate (assuming it is also SU(2)-symmetric).

III. TRIAL WAVE FUNCTIONS FOR GAPPED QSLS

Let us now illustrate how to build a trial gapped QSL wavefunction. We consider the specific case of a chiral spin liquid (CSL) [7], which is a ground state of SU(2) spin singlets that breaks parity and time-reversal. Even if this is a purely bosonic (spin) system, it is convenient to express spin operators in terms of fermion operators, $\vec{S} = \frac{1}{2} f^\dagger \vec{\sigma} f$.

The first step to build the wave function is to start with a model of fermions hopping on a lattice. For a CSL, we consider a square lattice with a π flux through every plaquette with the following mean-field Hamiltonian [2]

$$H = \sum_{\langle i,j \rangle} t_{ij} f_i^\dagger f_j + \mathrm{i} \sum_{\langle\langle i,j \rangle\rangle} \Delta_{ij} f_i^\dagger f_j \,, \tag{3}$$

where the first term is a hopping between nearest-neighbors (nn), while the second term represents an imaginary hopping along the diagonals (see Fig. 1 for more details)

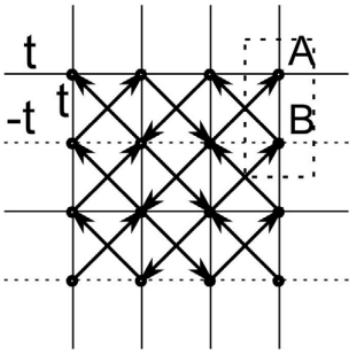

Figure 1: Square lattice on which the Hamiltonian (3) is defined. For nn hopping, $t_{ij} = t$ along the vertical direction. For full horizontal lines, $t_{ij} = t$, while $t_{ij} - t$ for dashed horizontal lines. For diagonal hoppings, $\Delta_{ij} = \mathrm{i}\,\Delta$ along the arrows and $\Delta_{ij} = -\mathrm{i}\,\Delta$ against the arrows. Figure taken from [3].

At this point, the ground state of the mean-field Hamiltonian (3), denoted $|\psi\rangle$, is simply an untangled fermionic product state without any long-range correlations. The candidate CSL ground state $|CSL\rangle$ is obtained by Gutzwiller projecting the mean-field state $|\psi\rangle$: $|CSL\rangle = P_G\,|\psi\rangle$, with $P_G = \prod_i (1 - n_{i,\uparrow} n_{i,\downarrow})\,|\psi\rangle$ and where $n_{i,\uparrow/\downarrow}$ is the number of up/-down spins on site i. This procedure has projected $|\psi\rangle$ to the Hilbert space subject to the constraint of a single electron/spin per site. Moreover, it has generated long-range entanglement between the spins, as it will be shown in the next section.

IV. MEASUREMENT OF THE TEE IN QSL WAVE FUNCTIONS

Now that we have an SU(2)-symmetric candidate ground state wave function for a chiral spin liquid, the goal is to determine if it has topological order. To do so, the strategy is to compute the topological entanglement entropy from $|CSL\rangle$.

A. Extraction of the TEE

A priori, extracting γ by computing the entanglement entropy of a single region A is rather challenging, since for a large boundary length (regime where terms of order $1/L$ can be neglected), the leading term proportional to L completely dominates over the term with γ in Eq. 2. However, by following the judicious trick from [4], which consists of considering the appropriate linear combination of entanglement entropies between three rectangular regions (represented in Fig. 2), the contribution of γ can be isolated as follows

$$\begin{aligned} -\gamma = {} & S_2(A) + S_2(B) + S_2(C) - S_2(AB) \\ & - S_2(AC) - S_2(BC) + S_2(ABC)\,. \end{aligned} \tag{4}$$

Note that the second Renyi entropy has been used instead of the von Neumann entropy, which will be justified in the next section.

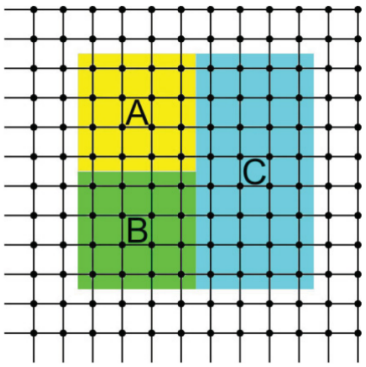

Figure 2: Illustration of the three regions where the second Renyi entropy S_2 is computed. The full system is a 12×12 square lattice with periodic boundary conditions, which means that the system lives on a 2-torus. The squares A and B have $L_A \times L_A$ sites, while the rectangle C has $L_A \times 2L_A$ sites. Figure taken from [3].

However, since $S_2(A) = S_2(B)$, $S_2(AB) = S_2(C)$ and $S_2(AC) = S_2(BC)$ [2], the expression for γ reduces to

$$\gamma = -2S_2(A) + 2S_2(AC) - 2S_2(ABC)\,. \tag{5}$$

B. Calculation of the TEE with a Variational Monte Carlo algorithm

The objective is now to compute the three entanglement entropies appearing on the RHS of Eq. (5), which is achieved by using a Variational Monte Carlo (VMC) algorithm. Naively, one might think that γ can be extracted from the von Neumann entropy ($n = 1$). However, it turns out that there are

no known Monte Carlo algorithms capable of computing S_1 directly. Nevertheless, a clever method developed in [8] allows to compute the second Renyi entropy, which is why S_2 is used to extract γ.

Let us illustrate how this technique works. Consider a system divided in two subregions A and B. The idea is to build two copies of a normalized pure state defined on the whole system

$$|\psi_i\rangle = \sum_{a_i b_i} C_{a_i b_i} |a_i\rangle |b_i\rangle \ , \tag{6}$$

where $i = 1, 2$ labels the copy, while $|a_i\rangle$ and $|b_i\rangle$ are respectively complete basis on A and B. Let us now consider a product state between the two independent copies. The SWAP operator over region A, SWAP_A, is then defined by acting between the two copies by exchanging the configurations in region A

$$\begin{aligned} &\text{SWAP}_A |\psi_1\rangle |\psi_2\rangle \\ &= \text{SWAP}_A \Bigg[\sum_{a_1 b_1} C_{a_1 b_1} |a_1\rangle |b_1\rangle \\ &\qquad \otimes \sum_{a_2 b_2} C_{a_2 b_2} |a_2\rangle |b_2\rangle \Bigg] \\ &= \sum_{a_1 b_1} C_{a_1 b_1} \sum_{a_2 b_2} C_{a_2 b_2} |a_2\rangle |b_1\rangle \otimes |a_1\rangle |b_2\rangle \ . \end{aligned} \tag{7}$$

Using this result, it can be shown (see Appendix A) that the second Renyi entropy is given by

$$S_2(\rho_A) = -\log \Big(\langle \text{SWAP}_A \rangle \Big) , \tag{8}$$

where the expectation value is taken with respect to the state $|\psi_1\rangle |\psi_2\rangle$.

By introducing the configurations $\alpha_1 = a_1, b_1$, $\alpha_2 = a_1, b_1$, $\beta_1 = a_2, b_1$ and $\beta_2 = a_1, b_2$, the expectation value of SWAP_A can be rewritten as [9]

$$\begin{aligned} \langle \text{SWAP}_A \rangle &= \sum_{\alpha_1 \alpha_2} \lambda(\alpha_1) \lambda(\alpha_2) f(\alpha_1, \alpha_2) \\ &= \langle \text{SWAP}_{A,\text{mod}} \rangle \langle \text{SWAP}_{A,\text{sign}} \rangle \ , \end{aligned} \tag{9}$$

where $\lambda(\alpha_i) = C_{\alpha_i} C^*_{\alpha_i}$ and $f(\alpha_1, \alpha_2) = \frac{C_{\beta_1} C_{\beta_2}}{C_{\alpha_1} C_{\alpha_2}}$. In the second line, the expectation value of the SWAP operator has been decomposed as a product of a modulus contribution and a phase contribution in order to deal with the sign of $f(\alpha_1, \alpha_2)$ which can be negative. See [9] for more details on the VMC algorithm and the subtleties due to potential negative signs.

C. Results for a chiral spin liquid

The Variational Monte Carlo technique presented above can then be applied to the CSL candidate wavefunction presented in Section III. Note that in order for the subleading ($\mathcal{O}(1/L)$) terms in Eq. (2) to be small, the boundary length L_A must be much bigger than the correlation length $\xi \sim \Delta^{-1}$. Hence, neat results are obtained by taking a large gap Δ [2].

As a consistency check, it is useful to compute the TEE for the unprojected (mean-field) wave function. For $L_A = 3$, it is found that $\gamma = -0.0008 \pm 0.0059$ [2], which is consistent with the expected result of $\gamma = 0$. Indeed, as mentioned previously, the unprojected wave function does not have topological order.

Moving-on to the Gutzwiller-projected wave function, it is found that $\gamma = 0.343 \pm 0.012$ and $\gamma = 0.344 \pm 0.043$ for $L_A = 3$ and $L_A = 4$ respectively [2], which is in great agreement with the expected value of $\gamma = \log \sqrt{2} \approx 0.347$. Indeed, the chiral spin liquid on the torus has two degenerate ground states and two types of abelian anyons [10], which means that $D = \sqrt{2}$.

Note that the VMC algorithm presented in this paper has been applied to other phases hosting topological order [2], such as a $\mathbb{Z}_2$ spin liquid or to fractional quantum Hall (Laughlin) states. The calculated TEEs are in good agreement with the theoretical predictions.

V. CONCLUSION

Gapped quantum spin liquids are exotic phases of matter lying beyond Landau's paradigm and hosting topological order. In this paper, a method to detect TO in candidate wave functions for gapped QSL using the topological entanglement entropy has been detailed. For the specific case of a chiral spin liquid on a torus, the calculated TEE with a Variational Monte Carlo algorithm yields values in excellent agreement with the expected results.

Finally, it is important to mention that the total quantum dimension D, obtained from γ, does not constitute a complete description of the topological order. Indeed, distinct phases can have the same D, while not having the same individual quantum dimensions d_a for their respective anyons [3]. However, it turns out that it is possible to extract the statistics and braiding of these anyons directly from the TEE. This work has been done in [11] and relies on the important fact that the topological entanglement entropy has a ground state dependence when subregions have boundaries that are not contractible.

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Appendix A: Algorithm to compute S_2

This appendix presents the derivation of Eq. (8). Using Eq. (7), the expectation value of the SWAP operator is

$$\begin{aligned}
\langle \mathrm{SWAP}_A \rangle &= \langle\psi_1|\langle\psi_2| \,\mathrm{SWAP}_A\, |\psi_1\rangle|\psi_2\rangle \\
&= \sum_{a_1b_1}\sum_{a_2b_2} C^*_{a_1b_1}C^*_{a_2b_2}\Big[\langle a_1|\langle b_1| \otimes \langle a_2|\langle b_2|\Big] \sum_{a_1'b_1'}\sum_{a_2'b_2'} C_{a_1'b_1'}C_{a_2'b_2'}\Big[|a_2'\rangle|b_1'\rangle \otimes |a_1'\rangle|b_2'\rangle\Big] \\
&= \sum_{a_1b_1}\sum_{a_2b_2}\sum_{a_1'b_1'}\sum_{a_2'b_2'} C^*_{a_1b_1}C^*_{a_2b_2}C_{a_1'b_1'}C_{a_2'b_2'}\,\delta_{a_1a_2'}\delta_{b_1b_1'}\delta_{a_2a_1'}\delta_{b_2b_2'} \\
&= \sum_{a_1a_2}\Big[\sum_{b_2} C_{a_1b_2}C^*_{a_2b_2}\Big]\Big[\sum_{b_1} C_{a_2b_1}C^*_{a_1b_1}\Big] .
\end{aligned} \tag{A1}$$

By defining the matrix elements of ρ_A as

$$(\rho_A)_{a_1a_2} = \sum_b C_{a_1b}C^*_{a_2b}\,, \tag{A2}$$

the expectation value becomes

$$\langle \mathrm{SWAP}_A \rangle = \sum_{a_1a_2} (\rho_A)_{a_1a_2}(\rho_A)_{a_2a_1} = \mathrm{Tr}(\rho_A^2)\,, \tag{A3}$$

which implies Eq. (8).