Area law and the s-sourcery framework

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We review the s-source framework proposed in [1] and expanded in [2,3,4]. In particular, we explore its relation and implications to gapped and gapless states. For gapped states, physical arguments are given to justify the area law for gapped states. For gapless states, we study a class of them that do satisfy the area law using local entanglement thermodynamics and relate it to the s-sourcery. Finally, we present examples of gapped and gapless RG circuits that implement the hierarchical growth given by the s-sourcery scheme.

INTRODUCTION

Entanglement entropy, a measure of the degree of entanglement between subsystems of a quantum system, has been the subject of intense research in the field of condensed matter and quantum information in recent years. The area law, which states that the entanglement entropy between two regions of a system is proportional to the area of the boundary between those regions, has been widely observed in gapped systems, where the energy gap between the ground state and the first excited state is nonzero. However, there exist no formal proof outside of Hastings seminal work in one dimension, at least for conventional gapped phases. Moreover, if we consider gapless states, there exist examples which do not follow the area law as conventional metals and conformal field theories (CFTs) in $1+1$ dimensions.

An interesting approach to try to answer some of these questions was proposed in [1] and is based on renormalization group (RG) ideas. In this paper, we review its framework and the consequences if phases of matter are RG s-source fixed points (to be defined in the next section). In particular, we will look the consequences for gapped phases and study its connection to a certain class of gapless systems. Lastly, we present an example of an $s=1$ source RG fixed point for gapless systems known as the square-root states.

S-SOURCE FRAMEWORK

Let's start with some context and definitions. We consider quantum systems whose Hilbert space is defined by associating smaller Hilbert spaces $\mathcal{H}_x$ to patches of space labelled by some coordinate x. Then couple them by the action of a local Hamiltonian $H=\sum_x H_x$ that couples locally patches x. Additionally, we will consider families of systems labelled by the system size L: H_L with groundstates $|\psi_L\rangle$.

We are interested in finding a finite-depth unitary that grows smaller groundstates $|\psi_L\rangle$ at size L plus ancillas into the groundstate $|\psi_{2L}\rangle$ at size $2L$. One naive expectation is to assume that we can make the groundstate ψ_{2L} out of only ancillas. Yet, only trivial phases of matter are adiabatically connected to the product state. Thus, it is not possible to find a gapped path from a product state to nontrivial phase using a finite-depth circuit.

Instead, we can think that using nontrivial groundstates at size L as seeds to feed into the unitary would help build the groundstate at larger size $2L$. This is where the s-source framework comes into play. A s-source RG fixed point in d-dimensions is a system whose groundstate on $(2L)^d$ sites can be made from s copies of the groundstate on L^d sites plus some unentangled ancillas using a quasilocal unitary. The value s is assumed to be the smallest one for which the procedure is possible.

Some examples are in order. Mean-field/trivial states is an $s=0$ RG fixed point. Toric code/$\mathbb{Z}_2$ gauge theory is an $s=1$ RG fixed point. All massive field theories are $s\leq 1$ RG fixed points since we can put the system into a background geometry of an expanding universe which maps it from size L to $2L$ without closing the gap. A more exotic state like Haah's cubic code is $s=2$ RG fixed point in $d=3$ dimensions. More examples can be found in [1].

We now present some of the implications of a state being a s-source RG fixed point.

Recursive entropy bounds

The entropy of a region of size $2R$ cannot be more than the sum of the entropies of the s regions R used plus an additional term coming from the unitary evolution,

$$sS(R)-k'R^{d-1}\leq S(2R)\leq sS(R)+kR^{d-1}, \quad (1)$$

where the quasilocal unitary can at most generate/remove area law entanglement leading to the upper/lower bound. This is a consequence of the small incremental entangling by a local Hamiltonian studied in class. Interestingly, by saturating the bound, we can show that any $s\leq 1$ fixed point in $d>1$ satisfies the area law theorem. Moreover, by making weak spectral assumptions, one can show that the possible range of s is $s\leq 2^{d-1}$.

Groundstate degeneracy

The groundstate degeneracy $G(L)$ of a s-source RG fixed on a d-torus of linear size L obeys

$$G(2L) = G(L)^s. \tag{2}$$

This is because we can construct different groundstates at scale $2L$ by choosing any of the $G(L)$ degenerate but locally indistinguishable groundstates from each of the s copies at scale L.

s is a property of the phase

The smallest possible value of s serves as a classification axis, quantifying the amount of entanglement in the groundstate. This is true since the adiabatic path connects any two representatives within the same phase.

RG circuit implies a MERA representation

As seen in class, the multiscale entanglement renormalization ansatz (MERA) is an important numerical method which allows for efficient computation of observables by organizing the entanglement in the state scale by scale. Numerical algorithms implementing the s-sourcery and MERA representations can be found in [4].

Entropy scaling in s-source fixed points

Let's assume that the recursion relation in Eqn. (1) is saturated. Then, the entropy at size R scales as

$$S(R) \sim R^{d-1} \sum_{l=0}^{\log R} \left(\frac{s}{2^{d-1}}\right)^l = \begin{cases} R^{d-1}, & s < 2^{d-1} \\ R^{d-1}\log(R), & s = 2^{d-1} \\ R^{d-1+\alpha}, & s = 2^{d-1+\alpha} \end{cases} \tag{3}$$

where as long as $s < 2^{d-1}$, the area law is obeyed.

We can find a formula for the groundstate degeneracy scaling as well. Taking the log of Eqn. (2), we find that

$$\log G(2L) \sim s^{\log(L)} \log G(2), \tag{4}$$

where $\log G(2)$ denotes the ground state degeneracy in a small system. The ground state degeneracy then scales as $s^{\log(R)}$. If $s = 2^{d-1}$, so that the area law is violated logarithmically, then the number of ground states need to grow as $\log G \sim L^{d-1}$. This fact will be important for the following subsection.

AREA LAW FOR GAPPED STATES

The fundamental assumption is that all stable gapped phases of matter are generalized s-source RG fixed points. And let's consider gapped phases without protected edge states, denoting as ρ_A the reduced density matrix of the groundstate in a subregion A. Let σ_A be the density matrix with maximal entropy locally in region A which is consistent with ρ_A, i.e. $\langle O_i\rangle_{\sigma_A} = \langle O_i\rangle_{\rho_A}$ for all local operators O_i supported in region A. Using variational methods, one can show that this state must be a local Gibbs state

$$\sigma_A = \frac{1}{Z} e^{-\tilde{H}_A} \tag{5}$$

where $\tilde{H}_A = \sum_{x\in A} g_x H_x$ and g_x are Lagrange multipliers such that its locally consistent with ρ_A as described above. We also know that since σ_A is the maximal entropy state that

$$S(\rho_A) \le S(\sigma_A). \tag{6}$$

Now, $\tilde{H}_A$ is locally gapped away from the boundary ∂A but it may have accidental edge states. However, we can repair this by perturbing with V localized near ∂A.

Let's bound $S(\sigma_A)$ by considering the thermal state of the fully gapped Hamiltonian $\tilde{H}_A + V$ labelled as σ'_A. Since we know that the thermal state minimizes the free energy then

$$\langle \tilde{H} + V\rangle_{\sigma'_A} - S(\sigma'_A) \le \langle \tilde{H} + V\rangle_{\sigma_A} - S(\sigma_A), \tag{7}$$

which after some rearrangements gives

$$S(\sigma_A) \le S(\sigma'_A) + [\langle \tilde{H} + V\rangle_{\sigma_A} - \langle \tilde{H} + V\rangle_{\sigma'_A}]. \tag{8}$$

The term in brackets is proportional to $|\partial A|$ since V is localized in the boundary and the expectation values of local H_x are approximately the same for σ_A and σ'_A.

This gives

$$S(\sigma_A) \le S(\sigma'_A) + \mathcal{O}(|\partial A|), \tag{9}$$

which we can further bound by remembering that $\tilde{H}_A + V$ is in the same phase as a gapped Hamiltonian on A with diverging local gap away from the boundary and thus bounded by the groundstate degeneracy G of $\tilde{H}_A + V$,

$$S(\rho_A) \le S(\sigma_A) \le \log(G) + \mathcal{O}(|\partial A|). \tag{10}$$

Using Eqn. (3) and (4), we see that violating the area law with logarithmic correction needs $s = 2^{d-1}$. This implies that the groundstate degeneracy grows only like $\log G \sim R^{d-1}$ as we have seen before. Yet, this violates the lower bound in (10) since we have $R^{d-1}\log R$ on the left and only R^{d-1} on the right. Hence, it must obey area law even with $s = 2^{d-1}$. If we further assume that the entropy is saturated, this implies that there are no gapped phases with $s = 2^{d-1}$. Nonetheless, We obtain a general law for gapped phases. Yay!

GAPLESS STATES AND S-SOURCERY

So far we have studied in the previous sections gapped states which satisfy the area law. And while this is believed to hold for all of these states, for gapless phases there exist exceptions. Most notably, conventional metals and CFTs in $1+1$ dimensions. Short-distance correlations usually lead to area law behavior. To violate this one would require some long-range correlations or many low-energy degrees of freedom (as it is the case for metals with Fermi surface). Instead, in this section, we will follow [2] and study a class of gapless states which, under certain thermodynamic assumptions, do follow the area law. We prove this by recasting the entanglement entropy problem into a local entanglement thermodynamics one, finding an area law bound and connecting the result with the s-sourcery framework.

Let's set up the problem. Consider a d-dimensional local Hamiltonian $H = \sum_X H_X$ with a $1/\mathrm{poly}(L)$ gap which supports scale invariance and whose groundstate is $|g\rangle$. Define the reduced density matrix of region A as $\rho_A = \mathrm{tr}_{\bar{A}} |g\rangle\langle g|$. Define also the state of maximum entropy σ_A which gives the same expectation values of all H_X inside A just as ρ_A. Since σ_A is the maximum entropy state it means that it should be a Gibbs state

$$\sigma_A = \frac{e^{-\sum_x H_x/T(x)}}{Z} \tag{11}$$

where $T(x)$ are again Lagrange multipliers such that $\langle H\rangle_{\rho_A} = \langle H\rangle_{\sigma_A}$, but also $S_{\rho_A} \leq S_{\sigma_A}$. Thus, we can bound the entropy of ρ_A using the entropy of σ_A.

Given that we are interested in the scaling with region size of the entropy and energy of σ_A, we can estimate them by using the local thermodynamic expressions

$$S_{\sigma_A} = -\,\mathrm{tr}(\sigma_A \log \sigma_A) \sim \int d^d x\, s(T(x)) \tag{12}$$

where $s(T)$ is the bulk thermodynamic entropy density. A similar expression can be found for the energy. These estimates are true if $\frac{\nabla T_x}{T_x} \ll \xi_x^{-1}$. Furthermore, let us specify the thermodynamic properties of the scale invariant state: the energy related to momentum as $\omega \sim k^z$ where z is the dynamical exponent and the entropy density related to temperature as $s(T) \sim T^{\frac{d-\theta}{z}}$ where θ is the hyperscaling violation exponent.

Let's proceed in estimating the integral by using a simple geometry as shown in Figure 1. Suppose the region A is a half space in d-dimensions with translation invariance in $d-1$-transverse dimensions to the boundary, each of size R with the remaining dimension having a width of ω. This means that $T(x)$ depends only on the distance to the boundary x ranging from a short-distance cutoff a (like the lattice spacing) to the UV cutoff (width ω of the half space). Since energy scales with momentum as $\omega \sim k^z$ and there's no other length scale in the problem, this determines the scaling of the temperature to be $T(x) \sim x^{-z}$. In fact, there exist two other solutions consistent with scale invariance which I won't focus in this paper but that are either relevant for frustration free Hamiltonians ($T=0$) or can be ruled out due to too much energy ($T=\infty$).

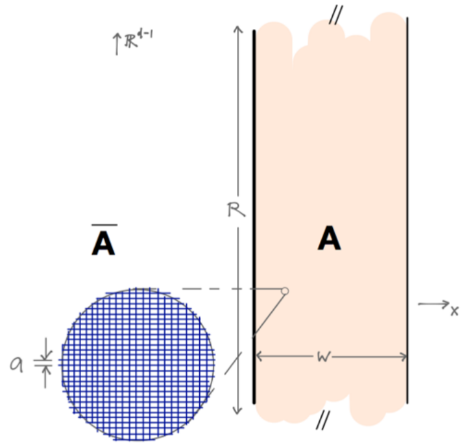

FIG. 1: Geometry used to derive the entanglement entropy bound. Figure taken from [2].

Assuming this, we can integrate the entropy density

$$S_{\sigma_A} \sim R^{d-1} \int_a^\omega dx\, x^{\theta-d} = R^{d-1} \begin{cases} \ln(\omega/a), & \theta = d-1 \\ \omega^{-d+\theta+1} - a^{-d+\theta+1}, & \text{other} \end{cases} \tag{13}$$

which converges for $d > \theta + 1$ and means that for values $\theta < d-1$ it obeys the area law and for $\theta = d-1$ it gives a logarithmic divergence with ω. Since we are upper bounding the entropy S_{ρ_A}, it does not follow that $\theta = d-1$ violates the area law. We also mention that we can do the integral over the energy density which leads to any local theory satisfying the assumptions to must have $z + d > 1 + \theta$.

We can compare this result with the s-sourcery framework. Suppose the scaling theory is an s-source RG fixed point. Then, it means that it obeys the entropy bound and the entropy scales as

$$S(R) \sim \frac{R^{d-1}}{a^{d-1}} \sum_{n=0}^{\log(\omega/a)} \left(\frac{s}{2^{d-1}}\right)^n = S_{\text{area}} + S_{\text{sub}} + \ldots \tag{14}$$

where the subleading term has the form

$$S_{\text{sub}} \sim \frac{R^{d-1}}{a^{d-1}} \frac{a^{d-1-\log(s)}}{\omega^{d-1-\log(s)}} \tag{15}$$

The coefficients in itself are not important as there may be discrepancies but the scaling behavior should be the same as the actual subsytem state ρ_A and its maximum entropy related state σ_A (since there's no phase transition in going from ρ_A to σ_A). Hence, comparing to Eqn. (13)

and demanding that the subleading terms match gives

$$s = 2^{\theta} \tag{16}$$

which establishes a deep connection between thermodynamics and entanglement.

RG CIRCUITS FOR GAPLESS STATES

An explicit class of examples of gapless $s = 1$ source fixed points arise from classical statistical models given by the so-called square-root states [3]. These are constructed as follows. Consider a classical model in d-dimensions with partition function

$$\mathcal{Z} = \sum_{s_i} e^{-\beta h(s_i)}, \tag{17}$$

where s_i represents a classical configuration, β is the inverse temperature and h is the classical Hamiltonian. We can build the square-root state living in a quantum many-body system in d-dimensions as

$$|\psi(h,\beta)\rangle = \sum_{s} \sqrt{\frac{e^{-\beta h(s)}}{\mathcal{Z}}}\, |\{s_i\}\rangle \,, \tag{18}$$

where now $|\{s_i\}\rangle$ are orthonormal states, β is a coupling and the Boltzmann weight determines the groundstate wavefunction. We can also see that correlations in the classical system are correlators of diagonal operators in the quantum system, meaning that critical points in the classical model become quantum critical points in the quantum model.

More importantly, we can use the real-space RG scheme to construct the quantum RG circuit. That is, turn the classical RG map into a unitary transformation which takes the state of size L plus ancillas to the state of size $2L$ on a larger geometry

$$U\,|\psi\rangle_L \otimes |0\ldots\rangle = |\psi\rangle_{2L} \otimes |0'\ldots\rangle \,. \tag{19}$$

In general, one may need to increase the on-site Hilbert space to be infinite for this to be an exact map but one can truncate the bond dimension if it's interested in an efficiently-contractible representation.

Let's take $1D$-Ising model as a case study. Its square-root state is

$$|\psi_I\rangle = \frac{1}{\sqrt{\mathcal{Z}}} \sum_{\{s_i\}} e^{\frac{\beta J}{2}\sum_i s_i s_{i+1}}\, |\{s_i\}\rangle \,, \tag{20}$$

which is the groundstate of the parent Hamiltonian

$$H = \sum_i (-X_i + e^{-\beta J Z_i (Z_{i-1}+Z_{i+1})}). \tag{21}$$

The reason why this model is chosen is because it's an exactly solvable example of an RG circuit as we shall see. This is because the real-space RG of the $1D$-classical model after each step has the same structure but with renormalized couplings. The renormalization procedure consists of tracing half of the degrees of freedom (say the odd spins) in the partition function $\mathcal{Z}$

$$\sum_{s_i} e^{\beta J (s_{i-1} s_i)} = 2\sqrt{\cosh(2\beta J)} \sum_{\text{even } \tilde{s}_i} e^{\tilde{\beta} J \tilde{s}_i \tilde{s}_i} \tag{22}$$

where the renormalized temperature is given by

$$\tilde{\beta} J = \frac{1}{2} \ln\cosh(2\beta J). \tag{23}$$

We then use this real-space RG transformation and explore the resulting RG transformation for the state and the Hamiltonian. Consider three neighboring sites with the spins $s_{i\pm1}$ being fixed

$$\left|\psi_i^{s_{i\pm1}}\right\rangle = \sum_{s_i} e^{\frac{\beta J}{2}(s_{i-1}s_i + s_i s_{i+1})}\, |s_{i-1}, s_i, s_{i+1}\rangle \,, \tag{24}$$

and apply the unitary U_i which disentangles the middle spin s_i with its neighbors

$$U_i \left|\psi_i^{s_{i\pm1}}\right\rangle = \sqrt{2}\cosh^{\frac{1}{4}}(2\beta J) e^{\frac{\tilde{\beta} J}{2} s_{i-1}s_{i+1}}\, |s_{i-1}s_{i+1}\rangle \otimes |{\rightarrow_i}\rangle \tag{25}$$

where $|{\rightarrow}\rangle$ is the eigenstate of X with positive eigenvalue. Then $U = \prod_{\text{odd } i} U_i$ converts all odd sites into product states and all even sites into a new Ising square root state with renormalized temperature $\tilde{\beta}$

$$U\,|\psi_I(\beta)\rangle_{2L} = |\psi_I(\tilde{\beta})\rangle_L \otimes \prod_{\text{odd } i} |{\rightarrow_i}\rangle \,. \tag{26}$$

Repeated applications of the RG circuit leads to a product state of all spin-right and the unitary to the identity. An sketch of the RG circuit is shown in Figure 2. We can apply this same circuit to obtain how the Hamiltonian transforms. The resultant Hamiltonian has the same form (21) but with renormalized $\tilde{\beta}$ and an overall constant $\frac{1}{2}e^{-2\tilde{\beta}J}(1 + e^{-2\tilde{\beta}J})$. We refer to the original paper for the explicit form of the unitary U_i and the calculations [3].

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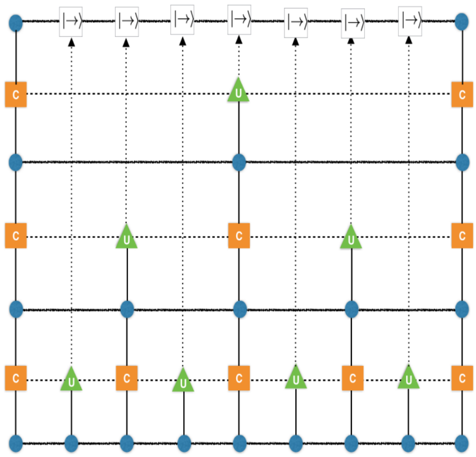

FIG. 2: RG circuit implementing the real-space RG of the $1D$-Ising model. Figure taken from [3].

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